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STRESS INTEGRATION OF THE DRUCKER-PRAGER MATERIAL MODEL WITH KINEMATIC HARDENING

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Abstract:

This paper presents a method for implicit stress integration of the Drucker-Prager material model with kinematic hardening. The stress integration of the material model is conducted using the incremental plasticity method, while the kinematic hardening of material is defined using nonlinear Armstrong-Frederick hardening. This type of granular material hardening occurs as a consequence of the cyclic loading effects, such as the seismic load. For this reason, this material model is used for the earthquake analysis in the soil mechanics. Yield surface of the material model changes its position under the cyclic loads in the stress space, whereas there is no change in the size of the yield surface in deviatoric plane. The developed algorithm of the material model has been implemented in the software package PAK.

Key words: Drucker-Prager, material model, Armstrong-Frederick, kinematic hardening, PAK

1. Introduction

Stress integration represents calculation of stress change during an incremental step, corresponding to strain increments in the step. The essence of the incremental integration of inelastic constitutive relations is to trace the history of material deformation. The stress integration is an important ingredient in the overall finite element inelastic analysis of structures. It is crucial that the integration algorithm accurately reproduces the material behavior since the mechanical response of the entire structure is directly dependent on this accuracy. The algorithm should be also computationally efficient because the stress integration is performed at all integration points. For general applications, this computational procedure should be robust, providing reliable results under all possible loading conditions. This paper presents a formulation of the computational algorithm for the Drucker-Prager (DP) constitutive model [1] with kinematic hardening feature, using incremental plasticity method (IPM) [2].

2. Elastic-plastic constitutive matrix using incremental plasticity method

Elastic-plastic constitutive models are described using elastic-plastic constitutive relations. In theory of incremental plasticity, stress is directly proportional to strain up to reaching yield stress. After reaching yield stress, strain increment can be divided into elastic and plastic part [3]:

$$d\mathbf{e} = d\mathbf{e}^E + d\mathbf{e}^P \tag{1}$$

Only elastic part of strain causes the stress change thus the stress increment can be formulated as:

$$d\boldsymbol{\sigma} = \mathbf{C}^E d\mathbf{e}^E \tag{2}$$

where $\mathbf{C}^E$ is elastic constitutive matrix. Substituting (1) in (2), the following is obtained:

$$d\boldsymbol{\sigma} = \mathbf{C}^E \left(d\mathbf{e} - d\mathbf{e}^P \right) \tag{3}$$

In the case of elastic-plastic constitutive models, yield function is the stress state function, therefore the increment of its change can be formulated as:

$$f = 0 \quad \text{and} \quad df = \frac{\partial f}{\partial \boldsymbol{\sigma}}^T d\boldsymbol{\sigma} = 0 \tag{4}$$

In incremental plasticity theory it is necessary that the failure function is in every time step less than or equal to zero (neutral loading condition).

Implicit stress integration implies the increment of plastic strain in the normal direction on the plastic potential surface, which can be formulated as:

$$d\mathbf{e}^P = d\lambda \frac{\partial g}{\partial \boldsymbol{\sigma}} \tag{5}$$

where $d\lambda$ is positive scalar, which is to be calculated, and plastic potential function g is the stress state function. Substituting the plastic strain increment (5) in (3) and using (4), it is obtained:

$$df = \frac{\partial f}{\partial \boldsymbol{\sigma}}^T \left(\mathbf{C}^E d\mathbf{e} - d\lambda \mathbf{C}^E \frac{\partial g}{\partial \boldsymbol{\sigma}} \right) = 0 \tag{6}$$

Plastic parameter $d\lambda$ can be calculated from equation (6) as:

$$d\lambda = \frac{\dfrac{\partial f}{\partial \boldsymbol{\sigma}}^T \mathbf{C}^E d\mathbf{e}}{\dfrac{\partial f}{\partial \boldsymbol{\sigma}}^T \mathbf{C}^E \dfrac{\partial g}{\partial \boldsymbol{\sigma}}} \tag{7}$$

Finally, using parameter $d\lambda$ from (7), stress increment $d\boldsymbol{\sigma}$ is obtained using (5) and (3) in the function of total strain increment:

$$d\boldsymbol{\sigma} = \mathbf{C}^{EP} d\mathbf{e} \tag{8}$$

where term $\mathbf{C}^{EP}$ represents elastic-plastic constitutive matrix.

3. Stress integration of the Drucker-Prager material model with kinematic hardening

Drucker-Prager material model is one of the oldest material models in soil mechanics [1, 4]. In the principal stress space, this surface represents a cone whose axis matches the space diagonal in the principal stresses space $\sigma_1, \sigma_2, \sigma_3$, as shown in Figure 1.

The yield surface equation of this model is a function of the stress state and defined as:

$$f = \alpha I_1 + \sqrt{J_{2D}} - k \tag{9}$$

In the case of non-associated yield condition the plastic potential surface is defined through the expression:

$$g = \beta I_1 + \sqrt{J_{2D}} \tag{10}$$

where I_1 represents the first stress invariant and J_{2D} represents second stress deviatoric invariant. Terms α , k and β represent material parameters of the material model which can be calculated indirectly using parameters of the Mohr-Coulomb model [5].

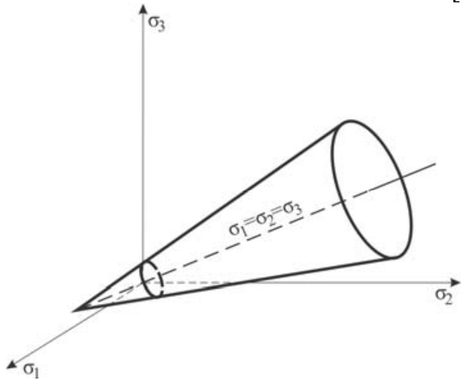

Figure 1 Drucker-Prager yield surface in principal stress space

In the analysis of the mechanical behaviour of soil exposed to cyclic loading such as earthquake, due to the effect of material hardening, models with kinematic hardening are often in use [6]. Yield surface of these models under the loads changes position in principal stresses space, whereas the size of the yield surface remains unchanged. Yield surface of the Drucker-Prager material model with kinematic hardening in deviatoric plane is shown in Figure 2.

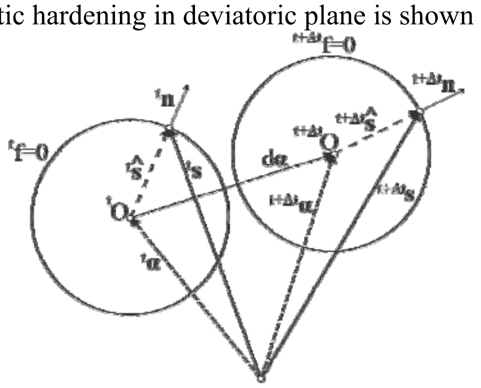

Figure 2 Yield surface in the deviatoric plain in case of kinematic hardening

Deviatoric stress in $t+\Delta t$ in case of kinematic hardening, according to Figure 2, can be expressed as:

$$ {}^{t+\Delta t}\mathbf{s} = {}^{t+\Delta t}\hat{\mathbf{s}} + {}^{t+\Delta t}\boldsymbol{\alpha} \tag{11} $$

where ${}^{t+\Delta t}\hat{\mathbf{s}}$ represents stress radius, whereas ${}^{t+\Delta t}\boldsymbol{\alpha}$ is the tensor internal variable termed back stress:

$$ {}^{t+\Delta t}\boldsymbol{\alpha} = {}^{t}\boldsymbol{\alpha} + d\boldsymbol{\alpha} \tag{12} $$

In this case, second stress invariant using stress radius has a form:

$$ J_{2D} = \frac{1}{2}\hat{\mathbf{s}}\cdot\hat{\mathbf{s}} = \frac{1}{2}(\mathbf{s}-\boldsymbol{\alpha})(\mathbf{s}-\boldsymbol{\alpha}) \tag{13} $$

Equation of the Drucker-Prager yield surface is a complex function of the stress, so using the chain rule, derivative of this function is:

$$ \frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{\partial f}{\partial I_1}\frac{\partial I_1}{\partial \boldsymbol{\sigma}} + \frac{\partial f}{\partial J_{2D}}\frac{\partial J_{2D}}{\partial \boldsymbol{\sigma}} \tag{14} $$

Derivative of the yield surface equation (9) with respect to stress invariant is:

$$\frac{\partial f}{\partial I_1} = \alpha \tag{15}$$

whereas his derivative with respect to second stress deviatoric invariant has a form:

$$\frac{\partial f}{\partial J_{2D}} = \frac{1}{2\sqrt{J_{2D}}} \tag{16}$$

Derivative of the second deviatoric stress invariant with respect to stress from (14) according to (13) can be calculated using the chain rule as:

$$\frac{\partial J_{2D}}{\partial \boldsymbol{\sigma}} = \frac{\partial J_{2D}}{\partial \hat{\mathbf{s}}} \frac{\partial \hat{\mathbf{s}}}{\partial \boldsymbol{\sigma}} \tag{17}$$

Second deviatoric stress invariant according to [7] using stress radius can be calculated as:

$$J_{2D} = \hat{s}_1\hat{s}_2 + \hat{s}_2\hat{s}_3 + \hat{s}_3\hat{s}_1 - \hat{s}_4^2 - \hat{s}_5^2 - \hat{s}_6^2 \tag{18}$$

so first term in equation (17) has a form:

$$\frac{\partial J_{2D}}{\partial \hat{\mathbf{s}}}^T = \left[\left(\hat{s}_2 + \hat{s}_3\right) \quad \left(\hat{s}_1 + \hat{s}_3\right) \quad \left(\hat{s}_3 + \hat{s}_1\right) \quad -2\hat{s}_4 \quad -2\hat{s}_5 \quad -2\hat{s}_6 \right] \tag{19}$$

Using equation (11), stress radius can be written as:

$${}^{t+\Delta t}\hat{\mathbf{s}} = {}^{t+\Delta t}\boldsymbol{\sigma} - \mathbf{m}\,{}^{t+\Delta t}\sigma_m - {}^{t+\Delta t}\boldsymbol{\alpha} \tag{20}$$

which implies second member of equation (17) in the form:

$$\frac{\partial \hat{\mathbf{s}}}{\partial \boldsymbol{\sigma}} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{21}$$

For mechanical behavior analysis of the granular materials exposed to non-proportional static or cyclic loading more than one type of kinematic hardening has been in use. The simplest type of kinematic hardening represents Drucker's linear kinematic hardening [8], whereas the increment of back stress $d\boldsymbol{\alpha}$ is collinear to the plastic strain increment and has the form:

$$d\boldsymbol{\alpha} = \frac{2}{3} C_1 d\mathbf{e}^P \tag{22}$$

where C_1 is material parameter, whereas $d\mathbf{e}^P$ represents plastic strain increment. However, significantly better definition of kinematic hardening has Armstrong-Fredrik hardening [9] due to the introduction of the member which represents dynamic relaxation. Back stress increment in this case has a form:

$$d\boldsymbol{\alpha} = \frac{2}{3} C_1 d\mathbf{e}^P - C_2\,{}^t\boldsymbol{\alpha} d\bar{e}^P \tag{23}$$

where C_2 represents material parameter, whereas $d\bar{e}^P$ is the equivalent plastic strain increment:

$$d\bar{e}^P = \sqrt{\frac{2}{3} d\mathbf{e}^P d\mathbf{e}^P} \tag{24}$$

Kinematic hardening can be implemented in the same way in other material models for mechanical behavior analysis of the granular materials. Algorithm for stress integration using Drucker-Prager material model with kinematic hardening is shown in Table 1.

Table 1 Algorithm for stress integration

A. Known: ${}^{t+\Delta t}\mathbf{e}$, ${}^{t}\mathbf{e}$, ${}^{t}\boldsymbol{\sigma}$, ${}^{t}\mathbf{e}^{p}$, ${}^{t}\boldsymbol{\alpha}$

B. Trial (elastic) solution:

$d\boldsymbol{\sigma} = \mathbf{C}^{E} d\mathbf{e}^{E} = \mathbf{C}^{E}\left({}^{t+\Delta t}\mathbf{e} - {}^{t}\mathbf{e}\right)$, ${}^{t+\Delta t}\boldsymbol{\sigma} = {}^{t}\boldsymbol{\sigma} + d\boldsymbol{\sigma}$, ${}^{t+\Delta t}\mathbf{s} = {}^{t+\Delta t}\boldsymbol{\sigma} - \mathbf{m}\,{}^{t+\Delta t}\sigma_{m}$,

B. Check the yield condition:

IF ($f \leq 0$) solution is elastic (GOTO **E**)

IF ($f >$) solution is elastic-plastic (CONTINUE)

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{\partial f}{\partial I_1}\frac{\partial I_1}{\partial \boldsymbol{\sigma}} + \frac{\partial f}{\partial J_{2D}}\frac{\partial J_{2D}}{\partial \boldsymbol{\sigma}}; \qquad \frac{\partial g}{\partial \boldsymbol{\sigma}} = \frac{\partial g}{\partial I_1}\frac{\partial I_1}{\partial \boldsymbol{\sigma}} + \frac{\partial g}{\partial J_{2D}}\frac{\partial J_{2D}}{\partial \boldsymbol{\sigma}}$$

$$d\lambda = \frac{\dfrac{\partial f}{\partial \boldsymbol{\sigma}}^{\mathbf{T}} \mathbf{C}^{E} d\mathbf{e}}{\dfrac{\partial f}{\partial \boldsymbol{\sigma}}^{\mathbf{T}} \mathbf{C}^{E} \dfrac{\partial g}{\partial \boldsymbol{\sigma}}}$$

C. Correction $d\lambda$ (local iterations):

$$d\mathbf{e}^{P} = d\lambda \frac{\partial g}{\partial \boldsymbol{\sigma}}, \quad d\overline{e}^{P} = \sqrt{\frac{2}{3} d\mathbf{e}^{P} d\mathbf{e}^{P}}, \quad d\boldsymbol{\alpha} = \frac{2}{3} C_1 d\mathbf{e}^{P} - C_2\, {}^{t}\boldsymbol{\alpha} d\overline{e}^{P}$$

$$d\mathbf{e}^{E} = d\mathbf{e} - d\mathbf{e}^{P}, \quad d\boldsymbol{\sigma} = \mathbf{C}^{E} d\mathbf{e}^{E}, \quad {}^{t+\Delta t}\boldsymbol{\sigma} = {}^{t}\boldsymbol{\sigma} + d\boldsymbol{\sigma}, \quad {}^{t+\Delta t}\boldsymbol{\alpha} = {}^{t}\boldsymbol{\alpha} + d\boldsymbol{\alpha}$$

D. IF ($ABS(f) \geq TOL$) go to **C** with new $d\lambda$:

${}^{t+\Delta t}\mathbf{e}^{P} = {}^{t}\mathbf{e}^{P} + d\mathbf{e}^{P}$

E. End: ${}^{t+\Delta t}\boldsymbol{\sigma}$, ${}^{t+\Delta t}\mathbf{e}^{P}$, ${}^{t+\Delta t}\boldsymbol{\alpha}$

4. Verification

Verification of the developed algorithm for implicit stress integration of Drucker-Prager material model with kinematic hardening is performed using triaxial test with cyclic loading. Unit size model with boundary conditions and loads is shown in Figure 3a, whereas the load functions are shown in Figure 3b.

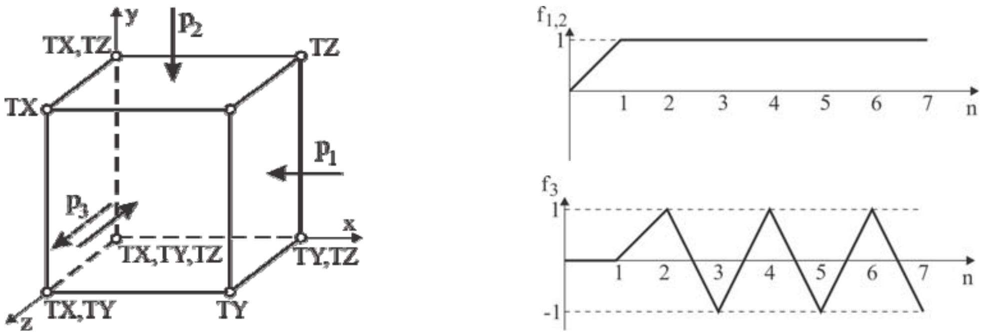

Figure 3 Model for cyclic loading simulation and load functions

Material parameters of the hardening model used in the cyclic loading analysis are shown in Table 1.

Table 2 Material parameters of the model

E [kPa]	ν	K [kPa]	α	C_1	C_2
100.0	0.25	10.0	0.0	20.0	1.4

Analysis results of the model loaded using cyclic load are shown in Figure 3, results for the used case models without kinematic hardening are shown in Figure 4a, whereas case of used model with Armstrong-Frederick kinematic hardening is shown in Figure 4b.

The analysis of the results presented in Figure 4 shows the effect of using Armstrong-Frederick kinematic hardening in case of cyclic loading of the specimen.

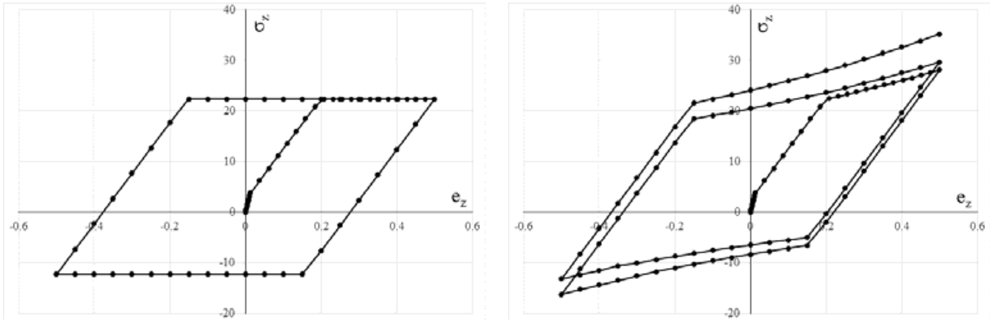

Figure 4 Axial stress vs. axial strain: a) withoud hardening, b) hardening model

In the case when the hardening of the model is not included, there is no change of stress during the cyclic loading. In the case of using the kinematic hardening effects during cyclic loading there is a change of the stress in the model, which is the main reason of using this model.

5. Conclusions

The results obtained using Drucker-Prager constitutive model with kinematic hardening are shown in the paper. The model for analyzing the mechanical behavior of materials exposed to the seismic loads is created through introduction of Armstrong-Frederick kinematic hardening in the existing model for the analysis of the mechanical behavior of granular materials. The advantage of the presented computational procedure is its general formulation which can be applied to similar material model. Verification example shows its hardening feature under cyclic loading as the main characteristics of the model.

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