Design of IIR lowpass differentiators based on second-order allpass filter application

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Abstract

The paper investigates the design of infinite-impulse response lowpass differentiators based on second-order allpass filter application. Specifications of the lowpass differentiator are first transformed into specifications of an equivalent second-order allpass filter. Unknown allpass filter coefficients are then determined such that relative passband magnitude response error of the corresponding lowpass differentiator is either minimized in the Chebyshev sense or maximally flat about some frequency in the passband. The main advantage of the proposed differentiators is their low group delay, which makes them suitable for real-time applications. Furthermore, the proposed equiripple design method exhibits fast convergence, while the maximally flat method is characterized by the simple closed-form expressions for the allpass filter coefficients values. Proposed lowpass differentiators are also compared with some of the existing infinite-impulse response lowpass differentiators.

Keywords: Digital lowpass differentiator, Digital allpass filter, Equiripple method, Maximally flat method

1 Introduction

Digital lowpass differentiators are required in various applications, among which physiological signals processing [7, 11, 20], axial strain calculation in ultrasound elastography [10], noise attenuation in control instrumentation [19], and dynamic phasor and frequency estimation [13]. Common to all these applications is that input signal's time derivative at low frequencies needs to be computed, while at the same time high-frequency noise suppressed. Like other types of filters, lowpass differentiators can be designed as finite- or infinite-impulse response (IIR) filters. In applications where perfectly linear passband phase response is not required, IIR lowpass differentiators are preferred over finite-impulse response lowpass differentiators due to lower filter order.

There are several approaches to the IIR lowpass differentiators design. The conventional one is to cascade the IIR fullband differentiator and the corresponding lowpass filter [4, 17]. Methods of the second approach are based on inversion of integrator's transfer function, followed by stabilization of unstable poles [1–3, 5], while methods of the third approach formulate the lowpass differentiators design problem as constrained [6, 12, 16] or unconstrained optimization problem [4, 21]. Utilization of the parallel allpass structure for the design of the IIR lowpass differentiators was recently proposed in [18].

A new parallel allpass structure for the IIR lowpass differentiators design is introduced in the paper, and two more allpass-based design methods are derived. A starting point for these design methods is the transformation of the lowpass differentiator's specifications into specifications of the corresponding second-order allpass filter. The first design method minimizes the relative error of the passband magnitude response. On the other hand, as the highest accuracy at frequencies about some frequency point is obtained by the maximally flat design [8, 9, 22], we also propose the maximally flat design method where relative error of the passband magnitude response is maximally flat about some frequency point.

The outline of the paper is as follows. In Sec. 2, an allpass-based IIR lowpass differentiator design problem is formulated. Proposed design methods and stability of obtained differentiators are discussed in Sec. 3, while design examples and comparison with the existing IIR lowpass differentiators are given in Sec. 4. Finally, concluding remarks are drawn in Sec. 5.

2 Problem formulation

Transfer function of the considered second-order IIR lowpass differentiators is assumed to be of the following form

$$H(z) = \gamma \frac{1 - D(z)}{2}, \quad \gamma > 0, \tag{1}$$

where $D(z)$ is the transfer function of the stable second-order allpass filter,

$$D(z) = \frac{b + az^{-1} + z^{-2}}{1 + az^{-1} + bz^{-2}}, \tag{2}$$

which phase response, denoted by $\phi(\omega)$, can be expressed in terms of the coefficients a and b as

$$\phi(\omega) = \arg\left\{D\left(e^{j\omega}\right)\right\} = -2\omega + 2\arctan\frac{a\sin\omega + b\sin(2\omega)}{1 + a\cos\omega + b\cos(2\omega)}. \tag{3}$$

By moving to the Fourier domain, ie. substituting $z = e^{j\omega}$ in Eq. 1, phase and magnitude responses of the considered IIR lowpass differentiators can be formulated as functions of the previously defined $\phi(\omega)$ as

$$\varphi(\omega) = \arg\left\{H\left(e^{j\omega}\right)\right\} = \frac{\phi(\omega) + \pi}{2}, \tag{4}$$

$$\left|H\left(e^{j\omega}\right)\right| = \gamma \cdot \sin\frac{-\phi(\omega)}{2}, \tag{5}$$

since the phase response $\phi(\omega)$ of the stable second-order allpass filter $D(z)$ is monotonically decreasing function for $\omega \in (0, \pi)$, while $\phi(0) = 0$ and $\phi(\pi) = -2\pi$ [15]. Therefore, $\left|H\left(e^{j0}\right)\right| = \left|H\left(e^{j\pi}\right)\right| = 0$ and $\varphi(0) = \pi/2$, regardless the values of the allpass filter coefficients a and b and the value of the parameter γ.

As the frequency response of an ideal IIR lowpass differentiator can be expressed as

$$\widetilde{H}\left(e^{j\omega}\right) = \begin{cases} j\omega e^{-j\omega\tau}, & \omega \le \omega_p \\ 0, & \omega > \omega_p \end{cases}, \tag{6}$$

where τ is the passband group delay, while ω_p is the passband-edge frequency, it can be concluded that magnitude response of the proposed IIR lowpass differentiators, Eq. 5, approximates the ideal one if the phase response of the stable allpass filter $D(z)$ approximates

$$\widetilde{\phi}(\omega) = \begin{cases} -2\arcsin\left(\omega\gamma^{-1}\right), & \omega \le \omega_p \\ -2\pi, & \omega > \omega_p \end{cases}, \tag{7}$$

where parameter γ obviously needs to satisfy $\gamma \ge \omega_p$.

Now, if $\phi(\omega)$ approximates $\widetilde{\phi}(\omega)$ in the passband, the phase response of the IIR lowpass differentiator, Eq. 4, approximates

$$\widetilde{\varphi}(\omega) = \frac{\widetilde{\phi}(\omega) + \pi}{2} = \begin{cases} \pi/2 - \arcsin\left(\omega\gamma^{-1}\right), & \omega \le \omega_p \\ -\pi/2, & \omega > \omega_p \end{cases}, \tag{8}$$

which is nearly-linear function of frequency in passband for $\gamma \gg \omega_p$. Denoting the ideal average passband group delay of the proposed IIR low-pass differentiators by

$$\widetilde{\tau} = \frac{\widetilde{\varphi}(0) - \widetilde{\varphi}(\omega_p)}{\omega_p} = \frac{1}{\omega_p}\arcsin\frac{\omega_p}{\gamma}, \tag{9}$$

passband phase response linearity error in ideal case can be expressed as

$$\widetilde{\varphi}(\omega) - \left(\frac{\pi}{2} - \omega\widetilde{\tau}\right) = \widetilde{\xi}\left(x = \frac{\omega}{\omega_p},\, y = \frac{\gamma}{\omega_p}\right) = x \arcsin\frac{1}{y} - \arcsin\frac{x}{y}. \tag{10}$$

Plot of $\max_{x\in[0,1]} \widetilde{\xi}(x, y)$ for $y \in [1, 3]$ is shown in Figure 1. As expected, when $y = \gamma/\omega_p$ increases, passband phase response linearity error decreases. On the

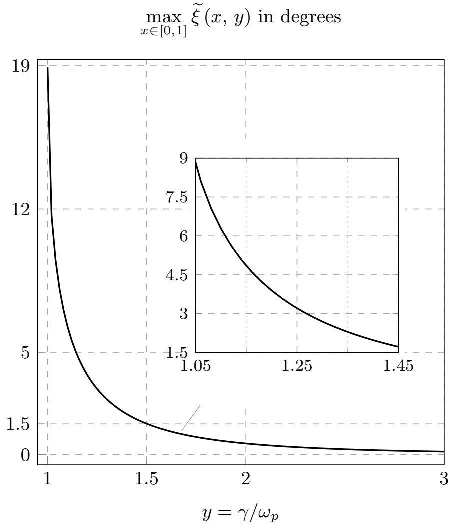

Fig. 1 Maximum passband phase response linearity error of the proposed IIR low-pass differentiators in ideal case.

other hand, as the maximum of the magnitude response, occurring at frequency $\omega = \omega^*$ where $\phi(\omega^*) = -\pi$, equals γ, note Eq. 5, γ should be close to ω_p (ie. $y = \gamma/\omega_p$ should be close to 1) to avoid excessive gain in the stopband.

As digital lowpass differentiators are typically designed such that relative error of the passband magnitude response is minimized [12], two algorithms for determination of the allpass filter coefficients a and b, for predetermined value of the parameter γ, are derived in this paper such that relative error of the passband magnitude response of the corresponding IIR lowpass differentiator

$$\zeta(\omega) = \omega^{-1}\Big(\left|H\left(e^{j\omega}\right)\right| - \omega\Big), \tag{11}$$

is either minimized in the Chebyshev sense or maximally flat about some frequency in the passband.

To circumvent the magnitude response error function, Eq. 11, being indeterminate for $\omega = 0$, sine term of the magnitude response, Eq. 5, is rewritten as

$$\sin\frac{-\phi\left(\omega\right)}{2}=\omega\cdot\lambda\left(\omega\right)=\omega\cdot\frac{\left(1-b\right)\operatorname{sinc}\omega}{\sqrt{1+a^2+b^2+2\big[a\left(1+b\right)\cos\omega+b\cos\left(2\omega\right)\big]}},\tag{12}$$

where

$$\operatorname{sinc}\omega=\begin{cases}1, & \omega=0\\ \sin\omega/\omega, & \omega\neq0\end{cases}.\tag{13}$$

Notations $\zeta\left(\omega,\,a,\,b\right)$, $\phi\left(\omega,\,a,\,b\right)$ and $\lambda\left(\omega,\,a,\,b\right)$ are used in the following text whenever dependence of $\zeta\left(\omega\right)$, $\phi\left(\omega\right)$ and $\lambda\left(\omega\right)$ on the allpass filter coefficients a and b needs to be emphasized.

Note that the transfer function of the proposed IIR lowpass differentiators, Eq. 1, requires at least 2 delays and 3 or 2 multiplications depending on the value of the parameter γ; if γ is adopted to equals the sum of a few power-of-two terms, only two multiplications are required.

3 Design methods

This section focuses on the methods for the design of IIR lowpass differentiators based on second-order allpass filter application. The first proposed design method, referred to as Equiripple method, minimizes the relative error of the passband magnitude response in the Chebyshev sense; while the second proposed design method, referred to as Maximally flat method, yields the maximally flat relative error of the passband magnitude response about some frequency $\omega=\omega_0$ in the passband.

3.1 Equiripple method

If relative error of the passband magnitude response of the proposed IIR lowpass differentiators, Eq. 11, is minimized in the Chebyshev sense, following is satisfied

$$\zeta\left(\widehat{\omega}_k\right)=\left(-1\right)^{p+k}\delta,\tag{14}$$

for $k=1,\,2,\,3$, where

$$\delta=\max_{\omega\leq\omega_p}\left|\zeta\left(\omega\right)\right|,\tag{15}$$

$0\leq\widehat{\omega}_1<\widehat{\omega}_2<\widehat{\omega}_3\leq\omega_p$ are frequencies where sign-alternating extremal values of $\zeta\left(\omega\right)$ occur in the passband, while p equals either 0 or 1 depending on whether local maximum or minimum of $\zeta\left(\omega\right)$ occurs at $\omega=\widehat{\omega}_1$.

Based on thorough inspection, it shows that p, figuring in Eq. 14, equals 0, ie. the local minimum of $\zeta\left(\omega\right)$ occurs first. Furthermore, it also shows that $\widehat{\omega}_1=0$ and $\widehat{\omega}_3=\omega_p$. Having this in mind, and rewriting Eq. 11 by means of Eq. 12 as

$$\zeta\left(\omega,\,a,\,b\right)=\gamma\lambda\left(\omega,\,a,\,b\right)-1,\tag{16}$$

Eq. 14 becomes

$$\begin{aligned} \gamma\lambda\left(0,\, a,\, b\right) &= 1-\delta, \\ \gamma\lambda\left(\widehat{\omega}_2,\, a,\, b\right) &= 1+\delta, \\ \gamma\lambda\left(\omega_p,\, a,\, b\right) &= 1-\delta. \end{aligned} \tag{17}$$

Now, after eliminating δ, previous system of equations reads

$$\begin{aligned} \lambda\left(0,\, a,\, b\right) - \lambda\left(\omega_p,\, a,\, b\right) &= 0, \\ \lambda\left(\widehat{\omega}_2,\, a,\, b\right) + \lambda\left(0,\, a,\, b\right) - 2\gamma^{-1} &= 0. \end{aligned} \tag{18}$$

As evidenced, obtained system of equations consists of 2 equations in 2 unknowns, if $\widehat{\omega}_2$ assumed known. However, as $\widehat{\omega}_2$ is not known in advance, the following exchange algorithm is proposed to obtain the unknown allpass filter coefficients a and b:

1. $t=0$. Determine a_0 and b_0 such that $\zeta\left(\omega,\, a_0,\, b_0\right)$ exhibits 3 sign-alternating extremal points in the closed interval $\left[0,\, \omega_p\right]$ by setting the magnitude response error equal to zero at two distinct frequencies:

$$\omega'_k = k\frac{\omega_p}{3}, \quad k = 1,\, 2, \tag{19}$$

   which is equivalent to

$$\phi\left(\omega'_k,\, a_0,\, b_0\right) = \widetilde{\phi}\left(\omega'_k\right) = -2\arcsin\left(\omega'_k\gamma^{-1}\right), \quad k=1,\,2. \tag{20}$$

   Now, as from Eq. 3 the phase response of the allpass filter $D\left(z\right)$ is related to its coefficients as

$$a\sin\frac{\phi\left(\omega\right)}{2} + b\sin\frac{\phi\left(\omega\right)-2\omega}{2} = -\sin\frac{\phi\left(\omega\right)+2\omega}{2}, \tag{21}$$

   unknown initial solution coefficients can be obtained as

$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} \omega'_1\gamma^{-1} & \sin\left(\omega'_1 + \arcsin\left(\omega'_1\gamma^{-1}\right)\right) \\ \omega'_2\gamma^{-1} & \sin\left(\omega'_2 + \arcsin\left(\omega'_2\gamma^{-1}\right)\right) \end{bmatrix}^{-1} \begin{bmatrix} \sin\left(\omega'_1 - \arcsin\left(\omega'_1\gamma^{-1}\right)\right) \\ \sin\left(\omega'_2 - \arcsin\left(\omega'_2\gamma^{-1}\right)\right) \end{bmatrix}. \tag{22}$$

2. Based on the known coefficients a_t and b_t, determine $0 < \widehat{\omega}_2^{(t)} < \omega_p$ where positive extremal value of the $\zeta\left(\omega,\, a_t,\, b_t\right)$ occurs; that is, the following nonlinear equation (derived by equating the first derivative of the right hand side of Eq. 16 to zero) needs to be solved

$$\begin{aligned} &\widehat{\omega}_2^{(t)}\left[a_t + \left(1+b_t\right)\cos\widehat{\omega}_2^{(t)}\right]\left(1 + a_t\cos\widehat{\omega}_2^{(t)} + b_t\right) \\ &-\sin\widehat{\omega}_2^{(t)}\left\{\left[a_t + \left(1+b_t\right)\cos\widehat{\omega}_2^{(t)}\right]^2 + \left(1-b_t\right)^2\sin^2\widehat{\omega}_2^{(t)}\right\} = 0. \end{aligned} \tag{23}$$

3. $t = t + 1$. Update the unknown coefficients a_t and b_t by solving the system of nonlinear equations

$$\begin{aligned} \lambda\left(0,\, a_t,\, b_t\right) - \lambda\left(\omega_p,\, a_t,\, b_t\right) &= 0, \\ \lambda\left(\widehat{\omega}_2^{(t-1)},\, a_t,\, b_t\right) + \lambda\left(0,\, a_t,\, b_t\right) - 2\gamma^{-1} &= 0, \end{aligned} \tag{24}$$

   note Eq. 18.
4. If $\max\left\{\left|a_{t-1} - a_t\right|,\, \left|b_{t-1} - b_t\right|\right\} \leq \Delta_{\text{tol}}$, where $\Delta_{\text{tol}} > 0$ is the prescribed tolerance, jump to the next step, otherwise proceed from the step 2.
5. The end of the algorithm. Unknown coefficients are $a = a_t$ and $b = b_t$.

Note that nonlinear equation (23) and system of nonlinear equations (24) can be solved in a variety of ways using different methods. For example, one can utilize the bisection method to determine $\widehat{\omega}_2^{(t)}$ in step 2, and the Powell's hybrid method [14] with the solution estimates for a_t and b_t set to a_{t-1} and b_{t-1}, respectively, in the step 3.

IIR lowpass differentiators derived using the proposed equiripple design method are not necessarily stable, which depends on the value of the parameter γ. If unstable, transfer function needs to be stabilized by reflecting the unstable pole(s) inside the unit circle, while magnitude needs to be compensated which is equivalent to decrease of the parameter γ. However, it shows that obtained differentiators are stable if $\gamma < \max\left\{0.78\pi,\, 1.04\omega_p\right\}$ for $\omega_p \in \left(0.05\pi,\, 0.85\pi\right)$.

3.2 Maximally flat method

Since there are only two unknown allpass filter coefficients, a and b, maximally flat design method starts from the following flatness conditions of $\zeta\left(\omega\right)$ at $\omega = \omega_0$,

$$\zeta\left(\omega_0,\, a,\, b\right) = 0, \tag{25}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}\omega}\zeta\left(\omega,\, a,\, b\right)\right|_{\omega=\omega_0} = 0. \tag{26}$$

As it shows that Eq. 26 is always satisfied for $\omega_0 = 0$, flatness conditions of $\zeta\left(\omega\right)$ at $\omega = \omega_0 = 0$ are given by Eq. 25 and

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\zeta\left(\omega,\, a,\, b\right)\right|_{\omega=0} = 0. \tag{27}$$

Therefore, for $\omega_0 = 0$ corresponding system of nonlinear equations obtained from Eqs. 25 and 27 by means of Eqs. 11 and 12 reads

$$\left(1 + a + b\right)^2 - \gamma^2\left(1 - b\right)^2 = 0, \tag{28}$$

$$-a^2 + ab + a - b^2 + 10b - 1 = 0, \tag{29}$$

which, after taking into account only a solution satisfying $b < 1$, yields

$$a = 2\frac{2\gamma^2 - 3}{\gamma^2 + 3\gamma + 3}, \tag{30}$$

$$b = \frac{\gamma^2 - 3\gamma + 3}{\gamma^2 + 3\gamma + 3}. \tag{31}$$

However, such obtained coefficients a and b are acceptable only if resulting IIR lowpass differentator's transfer function is stable, which shows to be true for $\gamma < 0.7797\pi$.

On the other hand, for $\omega_0 \neq 0$ following system of nonlinear equations is obtained from Eqs. 25 and 26 by means of Eqs. 11 and 12:

$$1 + a^2 + b^2 + 2\left[a\left(1+b\right)\cos\omega_0 + b\cos\left(2\omega_0\right)\right] - \gamma^2\left(1-b\right)^2 \mathrm{sinc}^2\omega_0 = 0, \tag{32}$$

$$\begin{aligned} &\omega_0\left[a + \left(1+b\right)\cos\omega_0\right]\left(1 + a\cos\omega_0 + b\right) \\ &-\sin\omega_0\left\{\left[a + \left(1+b\right)\cos\omega_0\right]^2 + \left(1-b\right)^2\sin^2\omega_0\right\} = 0, \end{aligned} \tag{33}$$

which, after eliminating solution that always results in unstable transfer functions, yields following solution for the unknown filter coefficients:

$$a = 2\frac{\left(1-p^2\right) - \mathrm{sinc}\left(2\omega_0\right)}{q}, \tag{34}$$

$$b = 1 - 2\frac{p\sqrt{1-p^2}}{q}\sin\omega_0, \tag{35}$$

where

$$p = \frac{\omega_0}{\gamma}, \tag{36}$$

$$q = \mathrm{sinc}\,\omega_0 - \left(1-p^2\right)\cos\omega_0 + p\sqrt{1-p^2}\sin\omega_0. \tag{37}$$

Again, coefficients a and b, determined by previous equations, are acceptable only if resulting IIR lowpass differentiator's transfer function is stable, which shows to be true for $\gamma < 0.7879\pi$ and every $\omega_0 < \gamma$, while for $\gamma > 0.7879\pi$, ω_0 should not be chosen too small. Note that Eq. 33 is of the same form as Eq. 23, which is expected as both equations are derived by equating the first derivative of the Eq. 16 to zero.

4 Design examples and comparison with the existing IIR lowpass differentiators

In this section design examples of the allpass-based IIR lowpass differentiators obtained by proposed design methods are discussed. All considered differentiators are stable as conditions derived in subsections 3.1 and 3.2 are satisfied. Proposed differentiators are also compared with the existing allpass-based differentiators from [18] and nearly-linear phase lowpass differentiators from [4].

It should be noted that transfer functions of second-order IIR lowpass differentiators from [1–3] can be also expressed in the form given by Eq. 1, however, corresponding allpass filters have double real poles, ie. one degree of freedom less than proposed lowpass differentiators, while passband-edge frequencies are not taken into account in the design of these differentiators.

To properly characterize considered IIR lowpass differentiators in the passband, the average passband group delay

$$\overline{\tau} = \frac{\varphi\left(0\right) - \varphi\left(\omega_p\right)}{\omega_p} = \frac{\pi}{2\omega_p} - \frac{\varphi\left(\omega_p\right)}{\omega_p}, \tag{38}$$

along with the corresponding passband phase response linearity error function in degrees [12]

$$\xi\left(\omega\right) = \frac{180}{\pi} \cdot \left[\varphi\left(\omega\right) - \left(\frac{\pi}{2} - \omega\overline{\tau}\right)\right], \tag{39}$$

as well as the maximum relative passband magnitude response error δ defined by Eq. 15, and the maximum absolute passband phase response linearity error

$$\eta = \max_{\omega \leq \omega_p} \left|\xi\left(\omega\right)\right|, \tag{40}$$

are used. In stopband, IIR lowpass differentiators are characterized by the average squared stopband magnitude response defined as

$$P_{sb} = \frac{1}{\pi - \omega_p} \int_{\omega_p}^{\pi} \left|H\left(e^{j\omega}\right)\right|^2 \mathrm{d}\omega. \tag{41}$$

4.1 IIR lowpass differentiators obtained by Equiripple method

The proposed equiripple method is utilized to design the IIR lowpass differentiators of various passband edges $\omega_p \in \left\{0.2\pi\left(1 + k/13\right) \,\middle|\, 0 \leq k \leq 39\right\}$ and $\gamma/\omega_p \in \{1,\, 1.02,\, 1.04\}$. Δ_{tol} in all examples equals 10^{-8}. Fast convergence is observed, ie. the needed number of iterations in the worst case equals 4. Plots of δ, η, P_{sb} and $\overline{\tau}$ as functions of ω_p, for three values of γ/ω_p, are shown in Figure 2. It can be observed that when the value of the parameter γ increases, the maximum relative passband magnitude response error δ and the maximum absolute passband phase response linearity error η decrease, note Figure 2A and Figure 2B. Further, as can be observed from Figure 2C, the average passband squared stopband magnitude response P_{sb} increases with the increase of the parameter γ. This conclusion is however expected, since, as mentioned before, the maximum of the magnitude response equals γ. On the other hand, the average passband group delay decreases when γ increases, Figure 2D. From the previous discussion, it follows that when δ and η decrease (which is achieved

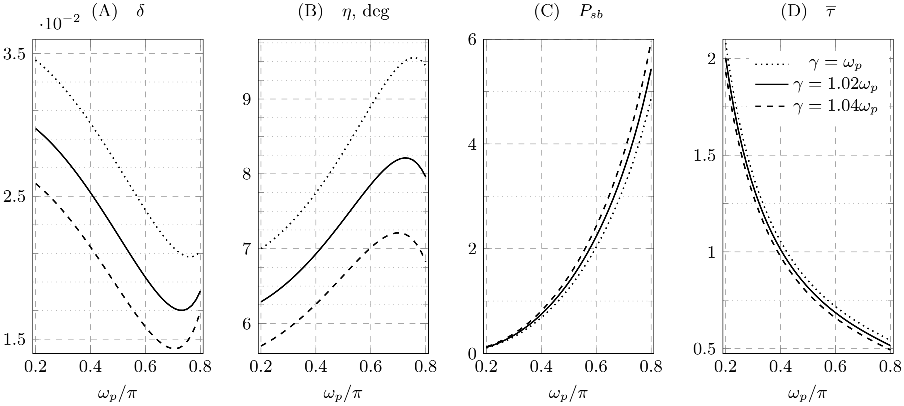

Fig. 2 IIR lowpass differentiators obtained by Equiripple method. (A) δ, (B) η, (C) P_{sb} and (D) $\overline{\tau}$ as functions of $\omega_p \in [0.2\pi,\ 0.8\pi]$ for three values of γ/ω_p.

by increasing the value of γ), P_{sb} increases, and vice versa. Therefore, γ should be chosen with great concern.

4.2 IIR lowpass differentiators obtained by Maximally flat method

The proposed maximally flat method is utilized to design the IIR lowpass differentiators of various passband edges, $\omega_p \in \{\pi\,(0.2 + 0.55k/39)\,|\,0 \leq k \leq 39\}$, $\omega_0/\omega_p \in \{0,\ 0.5,\ 0.65\}$ and $\gamma/\omega_p = 1.04$. Plots of δ, η, P_{sb} and $\overline{\tau}$ as functions of ω_p, for three values of ω_0/ω_p, are shown in Figure 3. It can be observed that when ω_0/ω_p increases, δ decreases, while η increases, note Figure 3A and Figure 3B. Furthermore, from Figures 2 and 3 it can be concluded that the passband phase response linearity error of differentiators designed using the maximally flat method, can be considerably lower compared to the differentiators designed using the equiripple method; this, however, comes at the price of somewhat higher maximum relative passband magnitude response error.

In order to get insight on how the choice of the parameter γ affects the performances of differentiators, maximally flat method is utilized to design the lowpass differentiators of various passband edges, $\omega_p \in \{\pi\,(0.2 + 0.55k/39)\,|\,0 \leq k \leq 39\}$, $\omega_0 = \omega_p/2$ and $\gamma/\omega_p \in \{1,\ 1.025,\ 1.05\}$. Plots of δ, η, P_{sb} and $\overline{\tau}$ as functions of ω_p, for three values of γ/ω_p, are shown in Figure 4. From these plots it can be observed that when γ increases, both δ and η decrease, while P_{sb} increases. This conclusion is expected as maximum of the magnitude response of the proposed differentiators equals γ. Therefore, as in the case of equiripple design method, the value of the parameter γ should be chosen such that a compromise is made between the stopband behavior and the passband phase response linearity and relative magnitude response errors.

It should be noted that transfer function of the second-order lowpass differentiator obtained by inverting the transfer function of the Simpson integrator

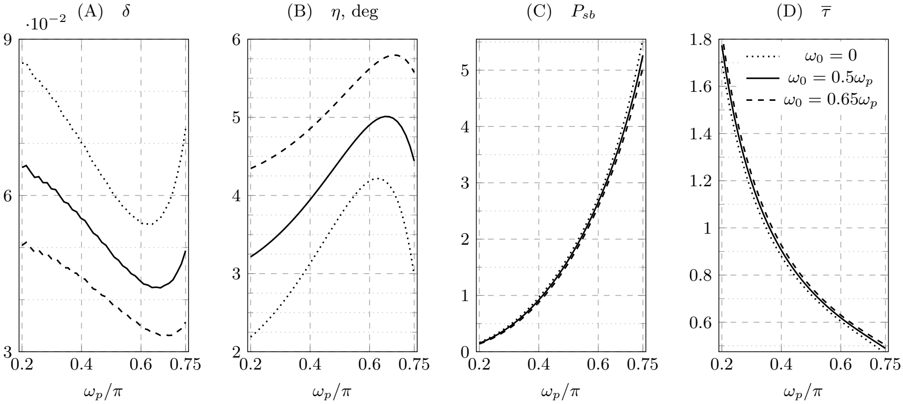

Fig. 3 IIR lowpass differentiators obtained by Maximally flat method. (A) δ, (B) η, (C) P_{sb} and (D) $\overline{\tau}$ as functions of $\omega_p \in [0.2\pi,\ 0.75\pi]$ for three values of ω_0/ω_p and $\gamma = 1.04\omega_p$.

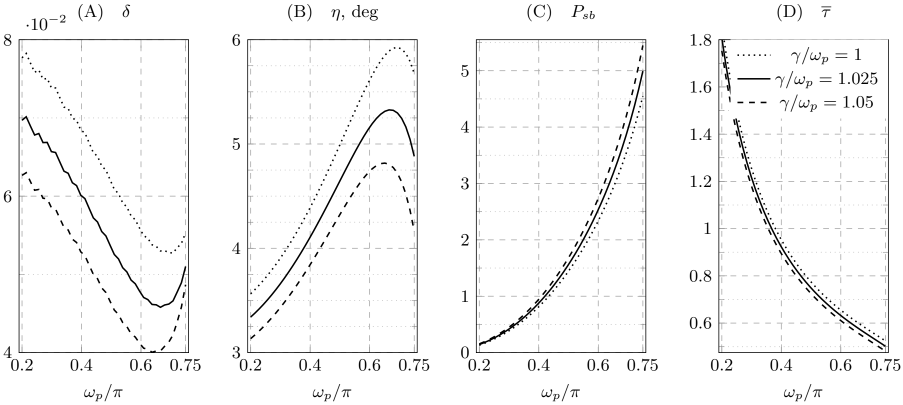

Fig. 4 IIR lowpass differentiators obtained by Maximally flat method. (A) δ, (B) η, (C) P_{sb} and (D) $\overline{\tau}$ as functions of $\omega_p \in [0.2\pi,\ 0.75\pi]$ for three values of γ/ω_p and $\omega_0 = \omega_p/2$.

followed by stabilization and magnitude compensation [2]

$$H_s(z) = \frac{3\left(1 - z^{-2}\right)}{3.7321\pi\left(1 + 0.2679z^{-1}\right)^2}, \tag{42}$$

can be obtained by the proposed maximally flat method if $\gamma = \max\left|H_s\left(e^{j\omega}\right)\right|$ and $\omega_0 = 0$.

4.3 Comparison with the existing allpass-based IIR lowpass differentiators

Utilization of the structure composed of two parallel allpass filters of the same orders, where one of the branches is a pure delay, for the design of the nearly-linear phase IIR lowpass differentiators was recently proposed in [18]. Transfer function of these differentiators is of the following form

$$H_1(z) = \gamma_1 \frac{A_L(z) - z^{-L}}{2}, \tag{43}$$

where $A_L(z)$ is the L-th order stable allpass filter.

To properly compare the allpass-based IIR lowpass differentiators from [18] with the proposed ones obtained by equiripple method, they have to be of the same order, ie. $L = 1$, while weighting function in [18] have to be adopted such that relative error of the passband magnitude response is minimized in the Chebyshev sense. Substituting $L = 1$ and $A_1(z) = \left(a_1 + z^{-1}\right) / \left(1 + a_1 z^{-1}\right)$ in Eq. 43, followed by some algebraic manipulations, transfer function of the second-order allpass-based IIR lowpass differentiators from [18] can be expressed as

$$H_1(z) = \frac{\gamma_1 a_1}{2}\left(1 - \frac{a_1 z^{-1} + z^{-2}}{1 + a_1 z^{-1}}\right). \tag{44}$$

On the other hand, if $b = 0$ in Eq. 2, transfer function given by Eq. 1 reduces to Eq. 44, where obviously $\gamma_1 = \gamma / a_1$, while $\gamma_1 a_1$ is the maximum of the magnitude response. Therefore, second-order IIR lowpass differentiators from [18] have one degree of freedom less than the proposed differentiators.

Lowpass differentiators from [18] are designed with $\gamma_1 = 2.25\omega_p$ for $\omega_p = 0.3\pi$ and $\gamma_1 = 2\omega_p$ for $\omega_p = 0.5\pi$, while proposed differentiators are designed by means of equiripple method with $\gamma = \gamma_1 a_1$. In this way, proposed and differentiators from [18] have equal maximums of the magnitude responses. Magnitude responses, relative passband magnitude response errors, and the passband phase response linearity errors of obtained allpass-based lowpass differentiators are given in Figure 5. As can be observed from these plots,

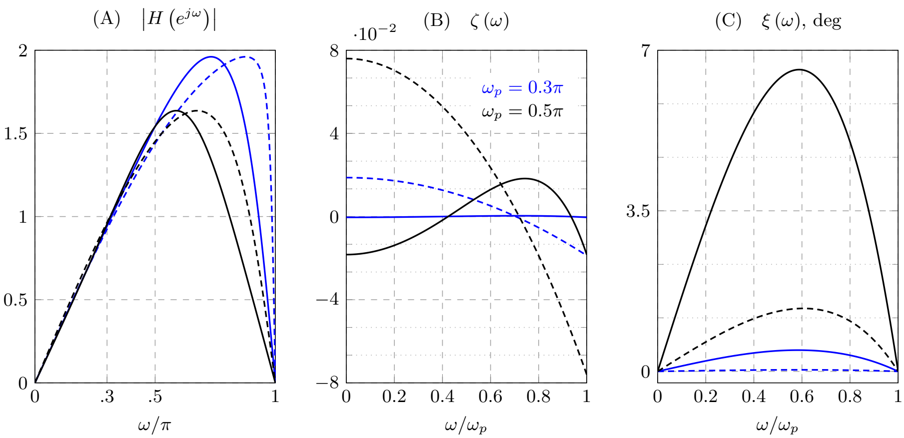

Fig. 5 Allpass-based IIR lowpass differentiators from [18] (dashed lines) and the proposed differentiators (solid lines): (A) Magnitude responses, (B) Relative passband magnitude response errors, (C) Passband phase response linearity errors.

proposed lowpass differentiators exhibit considerably lower relative passband magnitude response errors and better stopband behavior, but worse phase response linearity, compared to differentiators from [18]. Results of the comparison are summarized in Table 1. Denominator coefficients of the proposed

Table 1 Comparison between proposed and the IIR lowpass differentiators from [18]. Values that correspond to the lowpass differentiators from [18] are placed in brackets.

ω_p	$\overline{\tau}$	η	$\delta \cdot 100\%$	P_{sb}
0.3π	0.53 (0.52)	0.46 (0.03)	0.03 (1.87)	2.40 (2.62)
0.5π	0.78 (0.69)	6.58 (1.37)	1.83 (7.60)	1.48 (1.88)

lowpass differentiators, along with the corresponding values of the parameter γ, are given in Table 2.

Table 2 a, b and γ of the IIR lowpass differentiators competing the existing ones from [18].

ω_p	γ	a	b
0.3π	1.96065857	0.72158452	0.08097646
0.5π	1.63652373	0.30741904	0.13483000

4.4 Comparison with the nearly-linear phase lowpass differentiators from [4]

Two types of nearly-linear phase IIR lowpass differentiators are proposed in [4], unoptimized and optimized. The unoptimized differentiators are obtained by cascading the transfer function given by Eq. 42 with the third-order Chebyshev I lowpass filters having 0.1 dB ripple in the passband. On the other hand, optimized lowpass differentiators are derived from the unoptimized ones by altering the denominators coefficients such that resulting filters approximate a linear phase filters with magnitude response error also minimized [4].

As transfer function given by Eq. 42 can be obtained by means of the proposed maximally flat method if $\gamma = \max \left| H\left(e^{j\omega}\right) \right|$ and $\omega_0 = 0$, lowpass differentiators obtained by the proposed equiripple method are compared only to the optimized fifth-order nearly-linear phase IIR lowpass differentiators from [4] having the passband-edge frequencies equal to 0.3π, 0.4π and 0.5π. Values of the parameter γ of the proposed IIR lowpass differentiators are chosen as: $\gamma = 1$ for $\omega_p = 0.3\pi$, $\gamma = 1.03\omega_p$ for $\omega_p = 0.4\pi$, and $\gamma = \omega_p$ for $\omega_p = 0.5\pi$. Magnitude responses, relative passband magnitude response errors, and the passband phase response linearity errors of the proposed and the lowpass differentiators from [4] are shown in Figure 6, while results of the comparison are summarized in Table 3.

As proposed allpass-based differentiators are of second-order, comparison with the fifth-order lowpass differentiators may seem to be inappropriate. However, it is performed to illustrate that proposed second-order lowpass

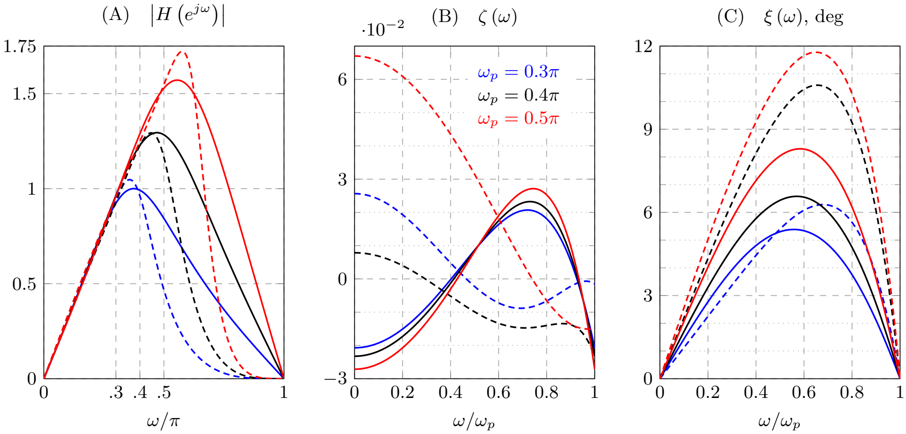

Fig. 6 Optimized IIR lowpass differentiators from [4] (dashed lines) and the proposed differentiators (solid lines), $\gamma = 1$ for $\omega_p = 0.3\pi$, $\gamma = 1.03\omega_p$ for $\omega_p = 0.4\pi$ and $\gamma = \omega_p$ for $\omega_p = 0.5\pi$: (A) Magnitude responses, (B) Relative passband magnitude response errors, (C) Passband phase response linearity errors.

Table 3 Comparison between proposed and the IIR lowpass differentiators from [4]. Values that correspond to the lowpass differentiators from [4] are placed in brackets.

ω_p	$\overline{\tau}$	η	$\delta \cdot 100\%$	P_{sb}
0.3π	1.25 (2.34)	5.38 (6.28)	2.07 (2.56)	0.41 (0.24)
0.4π	0.99 (1.98)	6.58 (10.59)	2.32 (2.34)	0.78 (0.41)
0.5π	0.85 (1.66)	8.29 (11.78)	2.71 (6.70)	1.24 (0.89)

differentiators can have lower passband phase response linearity and relative passband magnitude response errors compared to existing differentiators, note Figures 6B and 6C and Table 3. Furthermore, proposed differentiators also have lower average passband group delay, Table 3. On the other hand, optimized lowpass differentiators from [4] have narrower transition bands, Figure 6A, and lower average squared stopband magnitude response, Table 3, due to the higher filter order. Therefore, in applications where stopband behavior of the proposed second-order allpass differentiators is not acceptable, higher order differentiators have to be used.

The denominator coefficients of the proposed recursive lowpass differentiators, along with the corresponding values of the parameter γ, are given in Table 4.

Table 4 a, b and γ of the IIR lowpass differentiators competing the existing ones from [4].

ω_p	γ	a	b
0.3π	1	−0.47964673	0.24777529
0.4π	$1.03\,\omega_p$	−0.10745552	0.18604941
0.5π	ω_p	0.20015191	0.15851931

5 Conclusion

The novelty of the paper is the introduction of a new structure for the recursive lowpass differentiators design. This structure allows transformation of the differentiator specifications into specifications of an equivalent second-order allpass filter. Then, two methods for determination of the allpass filter coefficients are derived such that relative error of the passband magnitude response of the corresponding IIR lowpass differentiator is either minimized in the Chebyshev sense or maximally flat about some frequency.

While the equiripple design method exhibits fast convergence, the maximally flat method is characterized by the closed-form expressions for the coefficients values. It is shown that differentiators designed using maximally flat method can exhibit considerably lower passband phase response linearity errors compared to the differentiators designed using the equiripple method. An important feature common to both design methods is that value of the parameter γ needs to be chosen such that a compromise is made between the stopband behavior on one side, and the passband phase response linearity and relative magnitude response errors on other.

Results of comparison with the existing allpass-based recursive lowpass differentiators show that proposed differentiators exhibit lower relative passband magnitude response error and better stopband behavior, while comparison with some of the existing fifth-order differentiators shows that proposed differentiators can have lower passband phase response linearity and relative passband magnitude response errors. Finally, proposed recursive lowpass differentiators require only two or three multiplications and two delays which makes them suitable for real-time applications.

Declarations

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Conflict of interest

The author declares that he has no conflict of interest.

Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Ethics approval

Not applicable.

Consent to participate

Not applicable.

Consent to publication

Not applicable.

Code availability

Not applicable.

Authors' contributions

Not applicable.

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