Butterworth transfer function with the equalised group delay response in the maximally flat sense

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An effective method for the design of the continuous-time all-pass network for the low-pass filter group delay response equalisation is presented in this Letter. Since the method is derived from the maximally flat conditions at the origin of the group delay response's rational function, the cascade connection of the designed all-pass network and low-pass filter exhibits a constant group delay response at the origin in the flat sense. Illustrative examples regarding group delay equalisation of the Butterworth filters are given.

Introduction: A classical problem in the field of filter theory is the design of filters having constant group delay in the passband and at the same time steep magnitude selectivity. This problem can be often divided into two separate subproblems. The first one deals with the design of the filter with steep cut-off slope, which can be performed by utilising one of the many well-known low-pass approximation techniques. Since filter derived this way exhibits non-constant passband group delay response, the approximation of the additional all-pass equaliser is the second subproblem. Many researchers proposed different methods for the group delay equalisation in continuous- and discrete-time domain including, e.g. genetic algorithms based approach [1], adaptive filters [2], all-pass-based equalisation techniques [3] and the optimum equiripple delay frequency response method [4].

In this Letter, a novel approach to equalisation of the continuous-time low-pass filter group delay response is presented. The equalisation of the group delay response is carried out with the aid of the all-pass network, cascade-connected to the low-pass filter structure. The proposed method relies on a set of nonlinear equations, derived directly from the flatness conditions of the equalised group delay response at the origin, to obtain the unknown values of the all-pass network parameters.

Group delay equaliser design: Let us suppose that the group delay equalisation of the nth degree low-pass filter $G_n(s)$ is performed by cascading it with an mth degree all-pass network $A_m(s)$, whose magnitude response is equal to unity, but frequency varying phase response $\varphi_A(\omega)$. The transfer function of this cascade connection is $H(s) = G_n(s)A_m(s)$ or at real frequencies $s = \mathrm{j}\omega$

$$H(\mathrm{j}\omega) = |G_n(\mathrm{j}\omega)|\, \mathrm{e}^{\mathrm{j}[\varphi_G(\omega)+\varphi_A(\omega)]}, \tag{1}$$

where $\varphi_G(\omega)$ is the phase response of the low-pass filter.

Since the phase response of $H(\mathrm{j}\omega)$ equals to the sum of phase responses $\varphi_G(\omega)$ and $\varphi_A(\omega)$, and the group delay is obtained as the negative derivative of the phase response, one has that

$$\tau(\omega) = \tau_G(\omega) + \tau_A(\omega). \tag{2}$$

The group delay $\tau_G(\omega)$ of the low-pass filter $G_n(s)$ is an even, rational function in ω, which can be easily calculated by employing the following formula:

$$\tau_G(\omega) = \frac{1}{2}\left[\frac{1}{G_n(-s)}\frac{\mathrm{d}G_n(-s)}{\mathrm{d}s} - \frac{1}{G_n(s)}\frac{\mathrm{d}G_n(s)}{\mathrm{d}s}\right]_{s=\mathrm{j}\omega}. \tag{3}$$

On the other hand, since the transfer function of the mth degree all-pass filter $A_m(s)$ can be expressed as a cascade connection of the first-degree all-pass section (in terms of the natural frequency σ_0) and second-degree all-pass sections (in terms of the quality factors, Q_i, and corresponding natural frequencies, ω_{0i}), as follows:

$$A_m(s) = \left(\frac{-s+\sigma_0}{s+\sigma_0}\right)^{\mu} \prod_{i=1}^{k} \frac{s^2 - \dfrac{\omega_{0i}}{Q_i}s + \omega_{0i}^2}{s^2 + \dfrac{\omega_{0i}}{Q_i}s + \omega_{0i}^2}, \tag{4}$$

where $\mu = 0$ or 1 depending on the parity of $m = 2k + \mu$, its group delay response is given by

$$\tau_A(\omega) = \mu\frac{2\sigma_0}{\omega^2 + \sigma_0^2} + \sum_{i=1}^{k} \frac{2Q_i\omega_{0i}(\omega^2 + \omega_{0i}^2)}{Q_i^2\omega^4 + (1 - 2Q_i^2)\omega_{0i}^2\omega^2 + Q_i^2\omega_{0i}^4}. \tag{5}$$

The group delay of the equalised filter $\tau(\omega)$, obtained by putting (3) and (5) into (2), is an even rational polynomial function in ω

$$\tau(\omega) = \frac{b_0 + b_2\omega^2 + b_4\omega^4 + \cdots + b_{2(n+m-1)}\omega^{2(n+m-1)}}{a_0 + a_2\omega^2 + a_4\omega^4 + \cdots + a_{2(n+m)}\omega^{2(n+m)}} \tag{6}$$

with the coefficients being dependent on m all-pass network parameters. Ideally, the group delay response of the whole filter, given by (6), is constant in the maximally flat sense at the origin, i.e. its first $2(n+m) - 1$ derivatives with respect to the frequency equal to zero at $\omega = 0$. One way to accomplish this is by equating the ratio of the numerator and denominator coefficients multiplying the same powers of ω, i.e.

$$b_0/a_0 = b_2/a_2 = \cdots = b_{2(n+m-1)}/a_{2(n+m-1)}. \tag{7}$$

However, as there are only m unknown parameters for the all-pass network design, only $2m + 1$ derivatives of the group delay response given by (6) can be equal to zero. This results in m nonlinear equations with m unknowns for the group delay equaliser design

$$b_{2i}/a_{2i} - b_0/a_0 = 0, \quad \text{for } i = 1, 2, \ldots, m. \tag{8}$$

The Matlab Symbolic Toolbox can be used to solve the system of nonlinear equations (8) for the arbitrary degrees of both the low-pass filter and the group delay equaliser (i.e. all-pass network $A_m(s)$).

Design examples: The following examples illustrate the applicability of the proposed design method when Butterworth filter group delay equalisation is of interest. This is primarily because Butterworth filters are simple and commonly used in many analogue and digital signal processing applications. As the first example, the design of the third-degree group delay equaliser ($m = 3$) for the fourth-degree Butterworth filter ($n = 4$) is considered. Since the transfer function of the fourth-degree Butterworth filter in a symbolic form is given by

$$G_n(s) = \frac{1}{s^4 + \sqrt{2}\sqrt{\sqrt{2}+2}(s^3 + s) + (\sqrt{2}+2)s^2 + 1} \tag{9}$$

its group delay response, directly obtained by using (3), is written as

$$\tau_G(\omega) = \sqrt{2}\sqrt{\sqrt{2}+2}\cdot\frac{\omega^6 + (\sqrt{2}-1)(\omega^4 + \omega^2) + 1}{\omega^8 + 1}. \tag{10}$$

First, four numerator and denominator polynomials' coefficients of the whole system's group delay, in this case, see (6), are equal to

$$\begin{aligned}
b_0 =& h\, Q_1^2\sigma_0^2 + 2Q_1^2\sigma_0 + 2Q_1\sigma_0^2/\omega_{01},\\
b_2 =& h\left[Q_1^2 - \sigma_0^2(2Q_1^2 - 1)/\omega_{01}^2\right] + 2Q_1/\omega_{01}\\
&- 2\sigma_0(2Q_1^2 - 1)/\omega_{01}^2 + 2Q_1\sigma_0^2/\omega_{01}^3 + p\, Q_1^2 h\sigma_0^2,\\
b_4 =& h\left[Q_1^2\sigma_0^2/\omega_{01}^4 - (2Q_1^2 - 1)/\omega_{01}^2\right] + 2Q_1\omega_{01}^3 + 2Q_1^2\sigma_0/\omega_{01}^4\\
&+ p\, h\left[Q_1^2 - \sigma_0^2(2Q_1^2 - 1)/\omega_{01}^2 + Q_1^2\sigma_0^2\right],\\
b_6 =& p\, h\left[Q_1^2 - (\sigma_0^2 + 1)(2Q_1^2 - 1)/\omega_{01}^2 + Q_1^2\sigma_0^2/\omega_{01}^4\right]\\
&+ h\, Q_1^2(\sigma_0^2 + 1/\omega_{01}^4),
\end{aligned} \tag{11}$$

where $h = \sqrt{2}\sqrt{\sqrt{2}+2}$ and $p = \sqrt{2} - 1$, and

$$\begin{aligned}
a_0 &= Q_1^2\sigma_0^2, \quad a_2 = Q_1^2 - \sigma_0^2(2Q_1^2 - 1)/\omega_{01}^2,\\
a_6 &= Q_1^2/\omega_{01}^4, \quad a_4 = Q_1^2\sigma_0^2/\omega_{01}^4 - (2Q_1^2 - 1)/\omega_{01}^2.
\end{aligned} \tag{12}$$

Substituting (11) and (12) into (8), the following system of three nonlinear equations in the unknown biquad parameters Q_1, ω_{01} and σ_0 is obtained

$$\begin{aligned}
&c\, Q_1^3\sigma_0^3\omega_{01}^3 + (-1 + 3Q_1^2)\sigma_0^3 - Q_1^3\omega_{01}^3 = 0,\\
&c\, Q_1\sigma_0\omega_{01}^3\left[Q_1^2\sigma_0^2\omega_{01}^2 + Q_1^2\omega_{01}^2 - 2Q_1^2\sigma_0^2 + \sigma_0^2\right]\\
&\quad + 2Q_1^3\omega_{01}^3 + 3Q_1^2\sigma_0\omega_{01}^2 - Q_1^2\sigma_0^3 - Q_1\omega_{01}^3 - \sigma_0\omega_{01}^2 = 0,\\
&c\, \sigma_0\omega_{01}\Big[(1+\sqrt{2})Q_1^2\sigma_0^2\omega_{01}^4 - 2Q_1^2\sigma_0^2\omega_{01}^2 + Q_1^2\sigma_0^2 + Q_1^2\omega_{01}^4\\
&\quad - 2Q_1^2\omega_{01}^2 + \sigma_0^2\omega_{01}^2 + \omega_{01}^2\Big] - Q_1^2\omega_{01} - Q_1\sigma_0 = 0,
\end{aligned} \tag{13}$$

where $c = p\, h/2 = \left(1 - \sqrt{2}/2\right)\sqrt{\sqrt{2}+2}$.

After taking into consideration the stability and filter realisability conditions, the following solution is obtained: $\sigma_0 = 0.926892764227045$, $\omega_{01} = 0.999015631828311$ and $Q_1 = 0.625709073062524$.

Note that by putting $\sigma_0 = 0$ into (11) and (12), the coefficients a_0 and b_0 are equal to zero, while the remaining coefficients b_{2i} and a_{2i}, $i = 1, 2, 3$, can be used to design the second degree equaliser for the fourth degree Butterworth filter. The obtained system of two nonlinear equations yields the following solution: $Q_1 = 0.543397844468906$ and $\omega_{01} = 1.095461766679881$.

To illustrate this point, the group delay responses of the fourth-degree Butterworth filter and group delay-equalised Butterworth filters using second- and third-degree equalisers are shown in Fig. 1 left. Step responses of standard and group delay-equalised Butterworth filters are given in Fig. 1 right.

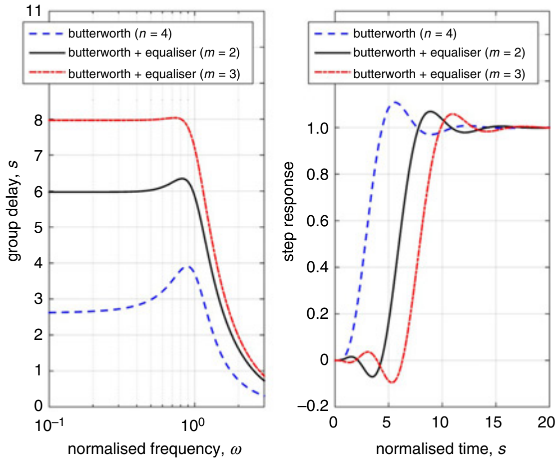

Fig. 1 *Group delay (left) and step (right) responses of fourth-degree Butterworth filter and group delay-equalised Butterworth filters using second- and third-degree delay equalisers*

As the second example, we investigated the group delay equalisation of the ninth-degree Butterworth filter using all-pass networks of degrees m from 2 to 9, obtained by utilising the proposed design method, all-pass networks containing one to five sections. The obtained quality factors, Q_i, and natural frequencies, ω_{0i}, $i = 1, 2, , \lfloor m/2 \rfloor$, for the biquad all-pass sections, and σ_0 of the first-degree all-pass sections, are given in Table 1.

Table 1: Particular parameters of biquads and first-degree all-pass sections of eight (m from 2 to 9) group delay equalisers (4) for ninth-degree Butterworth filter (if m is odd, σ_0 is denoted as ω_{0k}, $k = \lceil m/2 \rceil$)

m	k	i	ω_{0i}	Q_i	ε %
2	1	1	0.963504009246829	0.536093703423734	14.7990570
3	2	1	0.961269337258971	0.597915259429409	10.9110899
		2	0.897792816808062		
4	2	1	0.879996020982890	0.513953729797196	8.1434736
		2	0.964843302377879	0.665706871333869	
5	3	1	0.968656230697964	0.735283145515162	6.0891471
		2	0.875664576954717	0.542473192954504	
		3	0.850018973542872		
6	3	1	0.875610453463972	0.576924014652245	4.5435171
		2	0.971320993549263	0.805552707887652	
		3	0.837223073153313	0.507478285063141	
7	4	1	0.876405983124492	0.614331569298990	3.3695216
		2	0.831064915424474	0.524227879329719	
		3	0.972407847023535	0.876570380360233	
		4	0.817328398835526		
8	4	1	0.825962928941935	0.545982239794787	2.3939571
		2	0.803869831410650	0.504714507264969	
		3	0.875615282405391	0.654107415504115	
		4	0.971244143212340	0.950242528667801	
9	5	1	0.968602955311104	1.023263519925962	1.6383289
		2	0.798115157758117	0.515656364482048	
		3	0.824025442287063	0.569709959406258	
		4	0.874703878417463	0.693531488271415	
		5	0.790140705279375		

The maximum group delay relative errors of the equalised Butterworth filters,
$\varepsilon = (\tau_{max} - \tau_{min})/(\tau_{max} + \tau_{min}) \times 100$ %, are also given in the last column of Table 1.

The group delay responses of the equalised Butterworth filter for $n = 9$ with one up to five all-pass sections, whose biquad parameters are given in Table 1, are plotted in Fig. 2.

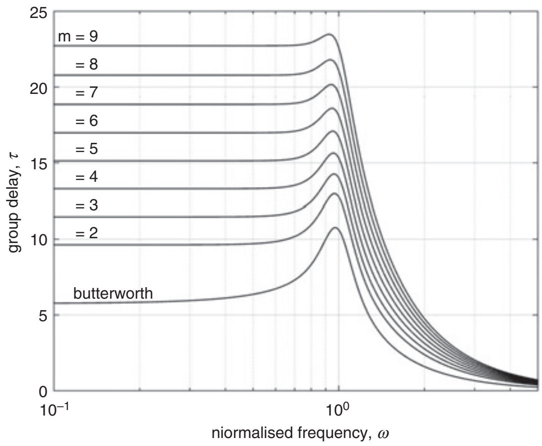

Fig. 2 *Group delay equalisation of a ninth-degree Butterworth filter; from bottom to top: original response and equalised responses using one to five all-pass sections*

Conclusion: In this Letter, a new design method for the group delay equalisers for the continuous-time filters is proposed. Unknown parameters characterising equaliser are obtained by solving a system of nonlinear equations, derived directly from the filter's group delay response flatness conditions at the origin. The obtained equalised filter exhibits a constant group delay response in the maximally flat sense at the origin, while the number of flatness can be arbitrarily specified by the degree of the group delay equaliser, i.e. the all-pass network. Finally, the proposed method can be extended towards the design of the group delay equalisers for other types of filters: both continuous- and discrete-time ones.

The proposed design method is suitable to be used for designing the group delay equalisers needed in applications such as impulse radio ultra-wideband wireless receivers, where exist necessity for the group delay response exhibiting maximally flat behaviour [5].

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One or more of the Figures in this Letter are available in colour online.

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References

1 Zaplatílek, K., Žiška, P., and Hájek, K.: 'Practice utilization of algorithms for analog filter group delay optimization', *Radioengineering*, 2007, **16**, (1), pp. 7–15
2 Farhang-Boroujeny, B., and Nooshfar, S.: 'Adaptive phase equalization using all-pass filters'. IEEE Int. Conf. on Communications, Denver, CO, 23–26 June 1991, pp. 1403–1407
3 Piskorowski, J., Kaszynski, R., Gutierrez de Anda, M.A., *et al.*: 'Group delay compensation and settling time minimization in continuous-time elliptic filters'. Proc. Mediterranean Electrotechnical Conf., Ajaccio, France, 5–7 May 2008, pp. 12–16
4 Quélhas, M.F., Petraglia, A., and Petraglia, M.R.: 'Efficient group delay equalization of discrete-time IIR filters'. 12th European Signal Processing Conf., Vienna, Austria, 6–10 September 2004, pp. 125–128
5 Zhou, L., Safarian, A., and Heydari, P.: 'CMOS wideband analogue delay stage', *Electron. Lett.*, 2006, **42**, (21), pp. 1213–1214