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RaDMI 2009

PROCEEDINGS

Volume 1

Editor:
Predrag V. Dašić

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PREFACE

The First Conference "Research and Development in Chemical and Mechanical Industry" - **RaDMI 2001** was held upon the initiative of Predrag Dašić and prof. dr Miroslav Radovanović in Kruševac from October 22-24, 2001.

Until now 8 conferences were realized. The conference accepted and published over 1.500 papers, from which 1.100 were from abroad from 40 various countries of the world. Total number of authors and coauthors is over 2.000. Papers of the 8th conferences were published in 13 proceedings in hard copy and 8 proceedings in electronic form (CD-ROM). Number of printed material was approximately 11.000 pages. Some papers from the 8th International conference RaDMI 2008 will be printed in special issue of international journal from SCI-E paper "Strojniški Vestnik – Journal of Mechanical Engineering" Vol. 55, no. 2 (2009) (Web site: http://en.sv-jme.eu/).

Ninth International Conference "Research and Development in Mechanical Industry" **RaDMI 2009** will be held on 16 – 19th September 2009 in Vrnjačka Banja, Serbia.

Topics of the Conference RaDMI 2009 are:

- **Plenary Session:** Invitation papers, with 13 papers;
- **Session A:** Research and development of manufacturing systems, tools and technologies, new materials and production design, with 46 papers;
- **Session B:** Transport systems and logistics, with 12 papers;
- **Session C:** Application of information technologies in mechanical engineering, with 25 papers;
- **Session D:** Quality management, ISO 9000, ISO 14000, TQM and management in mechanical engineering, with 48 papers;
- **Session E:** Application of mechanical engineering in other industrial fields, with 49 papers.

The aim of organizing the Conference is: animating scientists from the faculty and from institutes and experts from the industry and their connecting and collaboration, and exchanging the experiences and knowledge of domestic and foreign scientists and experts.

On behalf of the organizers, we would like to extend our thanks to all organizations and institutions that have supported the initiative to have this anniversary gathering organized. We would also like to extend our thanks to all authors and participants from abroad and from the country for contribution to this conference.

Vrnjačka Banja, September 2009.

CHAIRMAN OF ORGANIZING COMMITTEE
Dašić P.
Predrag Dašić, prof.

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CONTACT PROBLEM SOLUTION BY FINITE ELEMENT METHOD

Snezana D. Vulovic[1], **Miroslav M. Zivkovic**[2], **Nenad A. Grujovic**[3], **Radovan B. Slavkovic**[4]
[1] Faculty of Information Technology, Belgrade, SERBIA, vsneza@kg.ac.rs
[2] Faculty of Mechanical Engineering, Kragujevac, SERBIA, zile@kg.ac.rs
[3] Faculty of Mechanical Engineering, Kragujevac, SERBIA, gruja@kg.ac.rs
[4] Faculty of Mechanical Engineering, Kragujevac, SERBIA, radovan@kg.ac.rs

Summary: *In the paper a model for contact problem with friction, based on the penalty and Lagrange multiplier method, was described. As the configuration of two bodies coming into the contact is not a priori known, contact represents a nonlinear problem even when the continuum behaves as a linear elastic material. In this paper contact between two deformable bodies is considered as a general case. Presented approach, based on the Coulomb's frictional law, elasto-plastic tangential slip decomposition and consistent linearization. Due to the substantial similarity between friction and the classical elasto-plasticity [5,7], the constitutive model for friction was developed following the same formalism as in classical elasto-plasticity. The both models have been implemented into a version of the computational finite element program PAK [2].*

Keywords: *contact, penalty method, finite element*

1. INTRODUCTION

Contact mechanics has its application in many engineering problems, for example: the interaction between soil and foundations in civil engineering, general bearing problems as well as bolt and screw joints. Effective application of finite element contact solvers demands a high degree of experience since the general robustness and stability cannot be guaranteed. For this reason the development of more efficient, fast and stabile finite element contact discretizations is still a hot topic, especially due to the fact that engineering applications become more and more complex.

The aim of this paper is to provide framework for contact problems with friction, based on the penalty [2-4,7] and the Lagrange multiplier method [4,7]. The Lagrange multiplier method provides exact solutions but have additional degrees of freedom. The penalty formulation is purely geometrically based and therefore no additional degrees of freedom must be activated or inactivated but solution is dependent on introduced penalty factor. Numerical example is shown to demonstrate a possibility of applying the developed method in the analysis of finite deformation problems.

2. CONTACT KINEMATICS

As the configurations of two bodies coming into the contact are not a priori known, contact represents a nonlinear problem even when the continuum behaves as a linear elastic material. Two bodies are considered: $B^{(1)}$ and $B^{(2)}$, Fig. 1. Using a standard notation in contact mechanics, for each pair of contact surfaces, involved in the problem, we will define slave ($\Gamma_C^{(1)}$) and master surfaces ($\Gamma_C^{(2)}$), Fig. 1. The condition which must be satisfied is that any slave particle cannot penetrate the master surface. Let $\bar{\mathbf{x}}$ be the projection point of the current position of the slave node $\mathbf{x}^k$ onto current position of the master surface $\Gamma_C^{(2)}$, defined as

$$\frac{\mathbf{x}^k-\bar{\mathbf{x}}(\bar{\xi}^1,\bar{\xi}^2)}{\left\|\mathbf{x}^k-\bar{\mathbf{x}}(\bar{\xi}^1,\bar{\xi}^2)\right\|}\cdot\bar{\mathbf{a}}_\alpha(\bar{\xi}^1,\bar{\xi}^2)=0 \qquad (1)$$

where $\alpha = 1,2$ and $\bar{\mathbf{a}}_\alpha(\bar\xi^1, \bar\xi^2)$ are the tangent covariant base vectors at the point $\bar{\mathbf{x}}$. The definition of the projection point allows us to define the distance between any slave node and the master surface. The normal gap or the penetration g_N for slave node *k* is defined as the distance between current positions of this node to the master surface $\Gamma_C^{(2)}$:

$$g_N = (\mathbf{x}^k - \bar{\mathbf{x}}) \cdot \bar{\mathbf{n}} \tag{2}$$

where $\bar{\mathbf{n}}$ refers to the normal to the master face $\Gamma_C^{(2)}$ at point $\bar{\mathbf{x}}$ (Fig. 1). This gap (2) gives the non-penetration conditions as follows

$$g_N = 0 \quad \text{perfect contact;} \quad g_N > 0 \quad \text{no contact;} \quad g_N < 0 \quad \text{penetration} \tag{3}$$

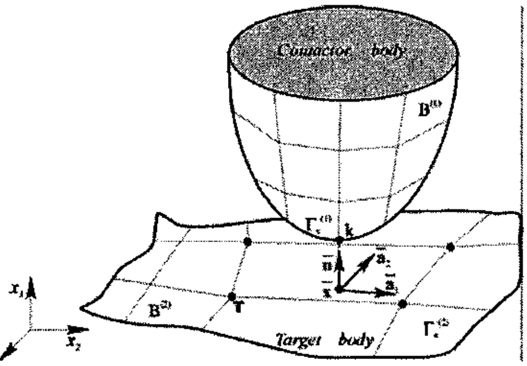

Figure 1: Geometry of the 3D node-to-segment contact element

The function (3) completely defines the contact kinematics for frictionless contact problem. If friction is modeled, tangential relative displacement must be introduced. In that case, the sliding path of the node $\mathbf{x}^k$ over the contact surface $\Gamma_C^{(2)}$ is described by total tangential relative displacement, in time interval from t_0 to t, as

$$g_T = \int_{t_0}^{t} \|\dot{\mathbf{g}}_T\| dt = \int_{t_0}^{t} \left\|\dot{\bar\xi}^\alpha \bar{\mathbf{a}}_\alpha\right\| dt = \int_{t_0}^{t} \sqrt{\dot{\bar\xi}^\alpha \dot{\bar\xi}^\beta a_{\alpha\beta}}\, dt \tag{4}$$

The time derivatives of parameter $\bar\xi^\alpha$ in equation (4) can be computed from the relation (1), [8]. In the geometrically linear case we obtain

$$\bar{a}_{\beta\alpha} \dot{\bar\xi}^\beta = \left[\dot{\mathbf{x}}^k - \dot{\bar{\mathbf{x}}}\right] \cdot \bar{\mathbf{a}}_\alpha = \dot{g}_{T\alpha} \tag{5}$$

where $\bar{a}_{\alpha\beta} = \bar{\mathbf{a}}_\alpha \cdot \bar{\mathbf{a}}_\beta$ is the metric tensor in point $\bar{\mathbf{x}}$ of the master surface $\Gamma_C^{(2)}$.

3. CONSTITUTIVE EQUATION FOR CONTACT INTERFACE

The stress acts on both surfaces obeying the action-reaction principle: $\bar{\mathbf{t}}(\bar\xi^1, \bar\xi^2) = -\mathbf{t}$ in the contact point $\bar{\mathbf{x}}$. A contact stress vector $\bar{\mathbf{t}}$ with respect to the current contact interface $\Gamma_C^{(2)}$ can be split into a normal and tangential part.

$$\bar{\mathbf{t}} = \bar{\mathbf{t}}_N + \bar{\mathbf{t}}_T = \bar{t}_N \bar{\mathbf{n}} + \bar{t}_{T\alpha} \bar{\mathbf{a}}^\alpha \tag{6}$$

where $\bar{\mathbf{a}}^\alpha$ is contravariant base vector. The tangential stress $\bar{t}_{T\alpha}$ is the zero in the case of frictionless contact. For contact one has the condition $\bar{t}_N < 0$. If there is not penetration between the bodies, then relations $g_N > 0$ and $\bar{t}_N = 0$ hold.

Using the penalty method for normal stress, constitutive equation can be formulated as

$$t_N = \varepsilon_N g_N \tag{7}$$

where ε_N is the normal penalty parameter.

In tangential direction a distinction is made between stick and slip. As long as no sliding between to bodies occurs, the tangential relative velocity is zero. If the velocity is zero, also the tangential relative displacement (5) is zero. A relative movement between two bodies occurs if the static friction resistance is overcome and the loading is large enough such that the sliding process can be kept. Therefore the relative sliding velocity, respectively the sliding displacement, shows in opposite direction to the friction force. With this the tangential stress vector is restricted as follows:

$$t_{T\alpha}^{sl} = -\mu \left| \mathbf{t}_N \right| \frac{\dot{g}_{T\alpha}^{sl}}{\left\| \dot{\mathbf{g}}_T^{sl} \right\|} \tag{8}$$

where μ is friction coefficient. In the simplest form of Coulomb's law (7), μ is constant and no distinction is made between static and sliding friction.

The tangential part is different for the stick and for the slip case. For stick a simple linear constitutive model can be used to describe the tangential stress

$$t_{T\alpha}^{stick} = \varepsilon_T g_{T\alpha} \tag{9}$$

where ε_T is the tangential penalty parameter. For slip the tangential stress given by the constitutive law for frictional sliding (8).

After the introduction of the stick and slip constraints, one needs an indicator to decide whether stick or slip actually take place. Therefore an indicator function

$$f = \left\| \mathbf{t}_T \right\| \text{-} \mu \left| t_N \right| \tag{10}$$

is evaluated, which respect the Coulomb's model for frictional interface law. In the equation (10) the first term is $\left\| \mathbf{t}_T \right\| = \sqrt{t_{T\alpha} \bar{a}^{\alpha\beta} t_{T\beta}}$. Then the following contact states can be distinguished:

$$f = \begin{cases} \left\| \mathbf{t}_T \right\| \text{-} \mu \left| t_N \right| \le 0 & \rightarrow \text{Stick} \\ \left\| \mathbf{t}_T \right\| \text{-} \mu \left| t_N \right| > 0 & \rightarrow \text{Slip} \end{cases} \tag{11}$$

A backward Euler integration scheme and return mapping strategy are used to integrate the friction equations (10), [4]. If a state of rod is assumed, the trial values of the tangential contact pressure vector $t_{T\alpha}$, and the indicator function f at load step *n+1* can be expressed in terms of their values at load step *n* as follows

$$t_{T\alpha\ n+1}^{trial} = t_{T\alpha\ n} + \varepsilon_T \Delta g_{T\alpha\ n+1} = t_{T\alpha\ n} + \varepsilon_T \bar{a}_{\alpha\beta} \Delta \xi_{n+1}^{\beta} \tag{12}$$

$$f_{Tn+1}^{trial} = \left\| \mathbf{t}_{Tn+1}^{trial} \right\| - \mu \left| t_{Nn+1} \right| \tag{13}$$

The return mapping is completed by

$$t_{T\alpha\ n+1} = \begin{cases} t_{T\alpha\ n+1}^{trial} & \text{if } f \le 0 \\ \mu \left| t_{Nn+1} \right| n_{T\alpha\ n+1}^{trial} & \text{if } f > 0 \end{cases} \tag{14}$$

with

$$n_{T\alpha\ n+1}^{trial} = \frac{t_{T\alpha\ n+1}^{trial}}{\left\| \mathbf{t}_{Tn+1}^{trial} \right\|} \tag{15}$$

4. EQUILIBRIUM EQUATION FOR BODIES IN CONTACT

When two bodies at time t are in contact the principle of virtual works can be written as (for a detailed legend of the symbols see [8])

$$\sum_{\alpha=1}^{2}\left(\int_{V^{(\alpha)}} \boldsymbol{\sigma}^{(\alpha)} : grad\,\delta\mathbf{u}^{(\alpha)} dV - \int_{V^{(\alpha)}} \rho^{(\alpha)}\left(\mathbf{b}^{(\alpha)} - \ddot{\mathbf{u}}^{(\alpha)}\right)\delta\mathbf{u}^{(\alpha)} dV - \int_{S_{\sigma}^{(\alpha)}} \boldsymbol{\sigma}^{(\alpha)}\cdot\mathbf{n}\cdot\delta\mathbf{u}^{(\alpha)} dA\right) - C_c = 0 \qquad (16)$$

where C_c is "contact contribution". For the Lagrange multiplier method for contact with friction contact contribution are formulated for stick as

$$C_c = \int_{S_C}\left(\lambda_N \delta g_N + \boldsymbol{\lambda}_T \cdot \delta\mathbf{g}_T\right) dA \qquad (17)$$

and for case of sliding

$$C_c = \int_{S_C}\left(\lambda_N \delta g_N + \mathbf{t}_T \cdot \delta\mathbf{g}_T\right) dA \qquad (18)$$

where δg_N and $\delta\mathbf{g}_T$ are variation of gap and tangential displacement; λ_N and $\boldsymbol{\lambda}_T$ are normal and tangential Lagrange multipliers and $\mathbf{t}_T$ is tangential stress vector which is determined from constitutive law for frictional slip. Note that the Lagrange multiplier λ_N can be identified as the contact stress t_N.
Contact contribution for the penalty method are formulated as follow

$$C_c = \int_{S_C}\left(\varepsilon_N g_N \delta g_N + \mathbf{t}_T \cdot \delta\mathbf{g}_T\right) dA \qquad (19)$$

5. FINITE ELEMENT FORMULATION

The virtual work of boundary nodes which are in contact is formulated for one slave node k:

$$C_c^k = F_N \delta g_N + \mathbf{F}_T \delta\mathbf{g}_T = t_N A_k \delta g_N + \mathbf{t}_T A_k \delta\mathbf{g}_T = t_N A_k \delta g_N + t_{T\alpha} A_k \delta\bar{\xi}^{\alpha} = \delta\mathbf{u}_c^T \mathbf{F}_c \qquad (20)$$

Here are: $F_N = t_N A_k$ the normal force; $F_{T\alpha} = t_{T\alpha} A_k$ the tangential force [8]; A_k the area of the contact element; $\mathbf{F}_c$ the contact force vector.
For the penalty method we define a displacement vector for the five-node contact elements (k, 1, 2, 3, 4)

$$\delta\mathbf{u}_c^T = \left\{\delta\mathbf{u}^k \quad \delta\mathbf{u}_1 \quad \delta\mathbf{u}_2 \quad \delta\mathbf{u}_3 \quad \delta\mathbf{u}_4\right\} \qquad (21)$$

and the vectors

$$\mathrm{N} = \begin{Bmatrix} \bar{\mathbf{n}} \\ -H_1\bar{\mathbf{n}} \\ -H_2\bar{\mathbf{n}} \\ -H_3\bar{\mathbf{n}} \\ -H_4\bar{\mathbf{n}} \end{Bmatrix} \qquad \mathbf{T}_\beta = \begin{Bmatrix} \bar{\mathbf{a}}_\beta \\ -H_1\bar{\mathbf{a}}_\beta \\ -H_2\bar{\mathbf{a}}_\beta \\ -H_3\bar{\mathbf{a}}_\beta \\ -H_4\bar{\mathbf{a}}_\beta \end{Bmatrix} \qquad \mathbf{D}^\alpha = \bar{a}^{\alpha\beta}\mathbf{T}_\beta \qquad (22)$$

Thus the contact force vector can be expressed with (26) for one slave node k which is in contact, by

$$\mathbf{F}_c = \left[F_N \mathbf{N} + F_{T\alpha}\mathbf{D}^\alpha\right] \qquad (23)$$

The contact forces F_N and $F_{T\alpha}$ in (27) can be obtain by multiplying the constitutive interfaces laws (15), (16) and (18) by the area of the contact element A_k.
In order to apply Newton's method for the solution system of equilibrium equation, a linearization of the contact contributions is necessary. In this paper, we do not state the linearization procedure for standard finite element formulation as well as the contact interface law for the normal and tangential part. It could be found in [4], [6].
In order to apply Newton's method for the solution nonlinear system of equilibrium equation (20), a linearization of the contact contributions is necessary. The linearization of the equation (25), for the infinitesimal theory, gives

$$\Delta t_N \delta g_N + \Delta t_{T\alpha} \cdot \delta \bar{\xi}^{\alpha} = \delta \mathbf{u}_c^T \mathbf{K}_c \Delta \mathbf{u}_c \tag{24}$$

where $\mathbf{K}_c$ is contact stiffness matrix of contact element. It is assumed that the contact area A_k is not changing significantly so the area A_k is contained within the penalty parameters. Tangent stiffness matrix for the normal contact is

$$\mathbf{K}_N = \varepsilon_N \mathbf{N}\mathbf{N}^T \tag{25}$$

Analogous to (29) we obtain symmetric tangent stiffness matrix for stick condition

$$\mathbf{K}_T^{stick} = \varepsilon_T \bar{a}_{\alpha\beta} \mathbf{D}^{\alpha} \mathbf{D}^{\beta T} \tag{26}$$

For slip condition get the tangent stiffness matrix is

$$\mathbf{K}_T^{slip} = \mu \varepsilon_N n_{T\alpha\ n+1}^{trial} \mathbf{D}^{\alpha} \mathbf{N}^T + \frac{\mu \varepsilon_N g_{Nn+1}}{\left\| \mathbf{t}_{Tn+1}^{trial} \right\|} \varepsilon_T \bar{a}_{\beta\gamma} \left[\delta_{\alpha}^{\beta} - n_{T\alpha\ n+1}^{trial} n_{T\ n+1}^{trial\ \beta} \right] \mathbf{D}^{\alpha} \mathbf{D}^{\gamma T} \tag{27}$$

The second term, the tangent matrix is non-symmetric. This is because the Coulomb's of friction can be viewed as a non-associative constitutive equation.
The linearization of the equations (21) and (22) give stiffness matrix for Lagrange multiplier method

$$\Delta t_N \delta g_N + \Delta t_{T\alpha} \cdot \delta \bar{\xi}^{\alpha} = \delta \mathbf{u}_c^T \mathbf{K}_{\lambda} \Delta \mathbf{u}_c \tag{28}$$

Detailed description of Lagrange multiplier method contact stiffness matrix is given in reference [6].
Finally, we obtain the global nonlinear finite element equation for penalty method

$$\mathbf{M}\ddot{\mathbf{U}} + [\mathbf{K} + \mathbf{K}_c]\mathbf{U} = \mathbf{F}(t) - \mathbf{F}_c \tag{29}$$

and for Lagrange multiplier method

$$\left\{ \begin{bmatrix} \mathbf{M} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{K} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{K}_{\lambda} \\ \mathbf{K}_{\lambda} & 0 \end{bmatrix} \right\} \begin{bmatrix} \Delta \mathbf{U} \\ \Delta \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{F}(t) \\ 0 \end{bmatrix} - \begin{bmatrix} \mathbf{F}_{\lambda} \\ g_N \bar{\mathbf{n}} \end{bmatrix} \tag{30}$$

where are: $\mathbf{M}$ mass matrix; $\mathbf{K}$ stiffness matrix and vector $\mathbf{F}(t)$ correspond to an external force. The contact force vector for the 3D contact elements for the Lagrange multiplier method is

$$\mathbf{F}_{\lambda}^T = \left[\boldsymbol{\lambda} \quad H_1 \boldsymbol{\lambda} \quad H_2 \boldsymbol{\lambda} \quad H_3 \boldsymbol{\lambda} \quad H_4 \boldsymbol{\lambda} \right] \tag{31}$$

6. EXAMPLE

A contact between two deformable bodies is considered [5]. The geometry data (in cm) and FE model are shown in Fig. 1 a). Plane strain elements are used. Both bodies have same material behaviors. Young's modulus is $E = 210\,\mathrm{N/cm^2}$, the Poisson's ratio equal to 0.3. Prescribed displacement of wedge-shaped body in vertical direction is given ($12 \times 0.05\mathrm{cm}$). The computation is performed for both Lagrange and penalty formulation ($\varepsilon_N = 1 \times 10^5$). Vertical stress field (σ_{yy}) for penalty formulation is shown in Fig. 1 b). In the table 6.1 values of vertical stress σ_{yy} in the node 1 using different software packages are shown.

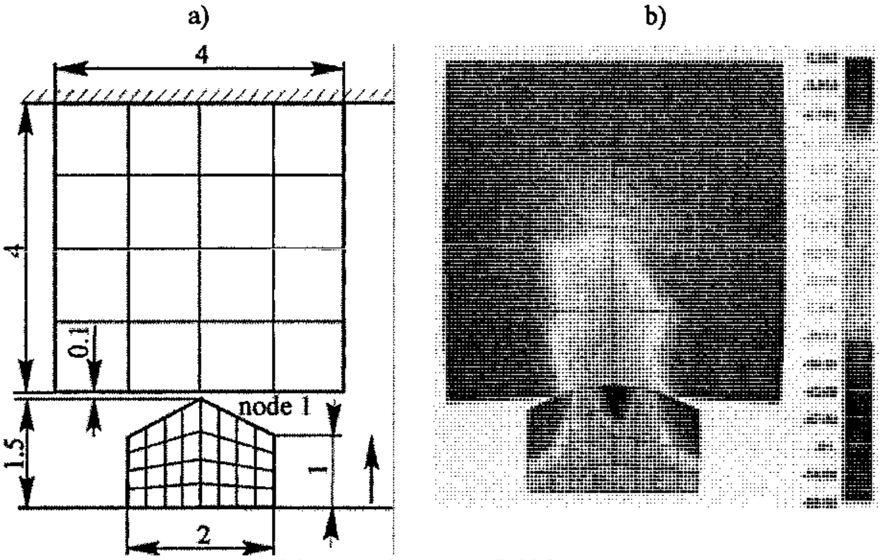

Figure 1: a) FE model; b) Vertical stress field for the penalty formulation

Table 1: Comparison of calculated stress

	NEiNastran	MSC.Marc	PAK - Lagrange	PAK- Penalty
Stress $\sigma_{yy}\left[\mathrm{N/cm^2}\right]$ at node 1	-58.06	-55.30	-59.04	-58.33

7. CONCLUSION

In the paper a model for three-dimensional contact problem with friction based on the penalty and Lagrange multiplier method was described. Using penalty method calculation time is less but results are strongly dependent on choice for a value of a penalty factor. The Lagrange multiplier method leads to exact solution but with more iterations and significant extension of a number of degrees of freedom i.e. equations and thus computational efficiency. The numerical example indicates a possibility of easy comparative simultaneous use of both developed procedures in the analysis of finite deformation problems within one computer code.

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REFERENCES

[1] Kojic M., R. Slavkovic, M. Zivkovic, N. Grujovic, The software packages PAK, Faculty of Mechanical Engineering of Kragujevac, Serbia.

[2] Laursen T.A., J.C. Simo, A continuum-based finite element formulation for the implicit solution of multibody, large deformation frictional contact problems, *Inter. J. Num. Meth. Eng.* 36 3451-3485, 1993.

[3] Peric D., R.J. Owen, Computational model for 3-D contact problems with friction based on the penalty method, *Inter. J. Num. Meth. Eng.* 35 1289-1309, 1992.

[4] Wriggers P., *Computational Contact Mechanics*, J. Wiley & Sons Ltd, West Sussex, England, 2002.

[5] Kojic M., K. J. Bathe, *Inelastic Analysis of Solids and Structures*, Springer, Berlin-Heidelberg, 2005.

[6] Grujovic N., Numerical solution of contact problems, Monograph, Faculty of Mech. Eng. Univ. of Kragujevac, Kragujevac, 2005.

[7] Zivkovic M., Nonlinear structural analysis, Monograph, Faculty of Mech. Eng. Univ. of Kragujevac, Kragujevac, 2006.

[8] Vulovic S., Numerical methods for solution contact problem based on the penalty method, Ph.D. Thesis, Faculty of Mech. Eng. Univ. of Kragujevac, Kragujevac, 2008.