ON THE IONIZED GAS BOUNDARY LAYER ADJACENT TO THE BODIES OF REVOLUTION IN THE CASE OF VARIABLE ELECTROCONDUCTIVITY

by

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Original scientific paper
DOI: 10.2298/TSCI111205223O

This paper studies the ionized gas i. e. air flow in an axisymmetrical boundary layer adjacent to the bodies of revolution. The contour of the body within the fluid is non-porous. The ionized gas flows under the conditions of equilibrium ionization. A concrete form of the electro-conductivity variation law has been assumed and studied here. Through transformation of variables and introduction of sets of parameters, V. N. Saljnikov's version of the general similarity method has been successfully applied. Generalized equations of axisymmetrical ionized gas boundary layer have been obtained and then numerically solved in a three-parametric localized approximation.

Key words: *ionized gas, boundary layer, body of revolution, general similarity method*

Introduction

This paper presents results of our investigations of the ionized gas flow in the boundary layer adjacent to the bodies of revolution. The ionized gas flows under the conditions of the so-called equilibrium ionization. The contour of the body within the fluid is non-porous.

The primary objective of this paper is to apply the general similarity method for a concrete form of the electroconductivity variation law.

The general similarity method was first used by Loitsianskii [1] and it was later improved by Saljnikov and Dallmann [2] – Saljnikov's version. Investigators of St. Petersburg School of Boundary Layer used this method to solve numerous problems of dissociated gas flow in the boundary layer. This method was also successfully applied to problems of planar dissociated gas boundary layer [3, 4]. Later, investigators of Belgrade School of Boundary Layer used Saljnikov's version of the boundary layer theory to solve practical problems of flow in the temperature and magnetohydrodynamics (MHD) boundary layers [5, 6]. This version was also used for solution of planar dissociated and ionized gas flow [7-11]. In this paper, Saljnikov's version of the general similarity method is applied.

Both versions of the general similarity method are based on usage of a momentum equation and introduction of corresponding sets of parameters [1]. The introduced form, magnetic and porosity parameters are called similarity parameters.

Mathematical model

It is well known that the ionized gas *i. e.* air flow is a multicomponental mixture of atoms, electrons, and positively charged oxygen and nitrogen ions [7, 12-14]. When the tempera-

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ture in the airflow is high enough, thermochemical equilibrium is established (equilibrium ionization).

One of the important properties of the ionized gas is its electroconductivity σ, which generally depends on the gas temperature *i. e.* enthalpy [15]. Analogous to MHD boundary layer [16], it is here assumed that the electroconductivity variation law can be written as:

$$\sigma = \sigma_0 \left(1 - \frac{u}{u_e} \right), \quad \sigma_0 = \text{const.} \tag{1}$$

Due to small thickness of the boundary layer, variation of the magnetic power at the cross-section of the boundary layer (in the direction of y-axis) can be ignored with this flow problem. Therefore, the component of the magnetic field power B_{my} is considered a function only of the longitudinal co-ordinate x [15]. In our investigations there was no need to define the function $B_m(x)$, since the so-called parametric solutions of the boundary layer equations are obtained here. Note that the expression $B_m(x) = \text{const.}/x^{1/2}$, given in the literature [15], has been used for this case of the ionized gas flow in the boundary layer. However, this expression is used exclusively to obtain auto-model solution of the boundary layer equations.

In the case when the ionized gas flow is under the effect of the outer magnetic field of the power $B_m = B_m(x)$, an electric flow is formed in the gas. The electric flow generates Lorentz force and Joule's heat. Due to these two effects, new terms appear in the ionized gas boundary layer equations. These terms cannot be found in the equations for homogenous unionized gas [15].

Therefore, the equation system of the steady laminar boundary layer adjacent to the bodies of revolution when the ionized gas flows in the magnetic field under the conditions of equilibrium ionization [7, 15, 17], can be written as:

$$\frac{\partial}{\partial x}(\rho u r^j) + \frac{\partial}{\partial y}(\rho v r^j) = 0, \quad (j = 1) \tag{2}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y} \right) - \sigma B_m^2 u \tag{3}$$

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = u \frac{\mathrm{d}p}{\mathrm{d}x} + \mu \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\partial}{\partial y}\left(\frac{\mu}{\Pr} \frac{\partial h}{\partial y} \right) + \sigma B_m^2 u^2 \tag{4}$$

$$\begin{aligned} &u = 0, \ v = 0, \ h = h_w = \text{const.} \quad \text{for} \quad y = 0 \\ &u \to u_e(x), \quad h \to h_e(x) \quad \text{for} \quad y \to \infty \end{aligned} \tag{5}$$

Based on the boundary conditions on the outer edge of the boundary layer (5) and on the electroconductivity variation law (1), we get $-\mathrm{d}p/\mathrm{d}x = \rho_e u_e/\mathrm{d}x$. Hence, the pressure $p(x)$ is eliminated from the eqs. (3) and (4). Then the governing equation system is brought to:

$$\frac{\partial}{\partial x}\left[\rho u \left(\frac{r}{L} \right)^j \right] + \frac{\partial}{\partial y}\left[\rho v \left(\frac{r}{L} \right)^j \right] = 0, \quad (L = \text{const.}, \ j = 1) \tag{2'}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \rho_e u_e \frac{\mathrm{d}u_e}{\mathrm{d}x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y} \right) - \sigma B_m^2 u \tag{3'}$$

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = -u \rho_e u_e \frac{\mathrm{d}u_e}{\mathrm{d}x} + \mu \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\partial}{\partial y}\left(\frac{\mu}{\Pr} \frac{\partial h}{\partial y} \right) + \sigma B_m^2 u^2 \tag{4'}$$

$$u=0,\ v=0,\ h=h_{\mathrm{w}}=\text{const.}\quad \text{for}\quad y=0$$
$$u\to u_{\mathrm{e}}(x),\quad h\to h_{\mathrm{e}}(x)\quad \text{for}\quad y\to\infty \tag{5'}$$

Equation (2) is a continuity equation of the axisymmetrical flow (j = 1) of the compressible fluid adjacent to the bodies of revolution. This equation is written in a form (2') more suitable for derivation of the momentum equation, where L is a characteristic length which can equal unity [4]. The eqs. (3') and (4') are dynamic and energy equation, respectively.

For the physical quantities in the governing equation system, notation usual in the boundary layer theory is used [13-15]. The radius $r(x)$ of the body of revolution is normal to the axis of revolution (fig. 1).

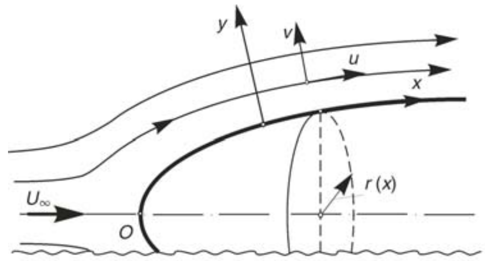

Figure 1. Gas flow adjacent to the body of revolution

The contour of the body is practically defined by the function $r(x)$. The boundary layer thickness $\delta(x)$ is assumed to be significantly smaller than the radius of the body of revolution ($\delta(x) \ll r(x)$). Therefore, it can be neglected, compared to $r(x)$ [14]. This assumption does not apply to long thin bodies.

Analogous to incompressible fluid flow [1], from the eqs. (2') and (3') by integration transversally to the boundary layer from $y = 0$ to $y \to \infty$, we come to the equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^\infty \rho u\left(\frac{r}{L}\right)^j (u_{\mathrm{e}}-u)\mathrm{d}y\right]=\left(\frac{r}{L}\right)^j \frac{\mathrm{d}u_{\mathrm{e}}}{\mathrm{d}x_{\mathrm{e}}}\int_0^\infty(\rho u-\rho_{\mathrm{e}}u_{\mathrm{e}})\mathrm{d}y+ \\ +\left(\frac{r}{L}\right)^j\left(\mu\frac{\partial u}{\partial y}\right)_{y=0}+\left(\frac{r}{L}\right)^j\sigma_0 B_{\mathrm{m}}^2\int_0^\infty\left(1-\frac{u}{u_{\mathrm{e}}}\right)u\mathrm{d}y \tag{6}$$

In order to solve the integrals, new variables are introduced in the form of the following transformations:

$$s(x)=\frac{1}{\rho_0\mu_0}\int_0^x \rho_{\mathrm{w}}\mu_{\mathrm{w}}\left(\frac{r}{L}\right)^{2j}\mathrm{d}x,\quad z(x,y)=\left(\frac{r}{L}\right)^j\int_0^y\frac{\rho}{\rho_0}\mathrm{d}y \tag{7}$$

The transformations (7) for j = 0 were used in numerous scientific papers [3, 18]. Here, ρ_0 and $\mu_0=\rho_0\nu_0$ denote known values of the density and dynamic (kinematic) viscosity at a certain point of the boundary layer, while ρ_{w} and μ_{w} stand for their known values on the wall of the body of revolution.

Changing the variables, *i. e.* solving the integrals in eq. (6) using the variables (7), the momentum equation is relatively easily obtained. This equation can be written in its three forms:

$$\frac{\mathrm{d}Z^{**}}{\mathrm{d}s}=\frac{F_{\mathrm{m}}}{u_{\mathrm{e}}},\quad \frac{\mathrm{d}f}{\mathrm{d}s}=\frac{u_{\mathrm{e}}'}{u_{\mathrm{e}}}F_{\mathrm{m}}+\frac{u_{\mathrm{e}}''}{u_{\mathrm{e}}'}f,\quad \frac{\Delta^{**\prime}}{\Delta^{**}}=\frac{u_{\mathrm{e}}'}{u_{\mathrm{e}}}\frac{F_{\mathrm{m}}}{2f} \tag{8}$$

where (') stands for a derivative per a longitudinal variable s.

Equation (8) are formally the same as the momentum equations for the planar [3] and axisymmetrical boundary layer [19].

In order to obtain the momentum equation the following quantities need to be defined: the form parameter $f(s)$, magnetic parameter $g(s)$, characteristic function of the boundary layer

F_m, conditional displacement thickness $\Delta^*(s)$, conditional momentum loss thickness $\Delta^{**}(s)$, conditional thickness $\Delta_1^{**}(s)$, and non-dimensional friction function $\zeta(s)$. They are defined by the expressions:

$$f(s)=\frac{u_\mathrm{e}'\Delta^{**2}}{\nu}=u_\mathrm{e}'Z^{**}=f_1(s);\quad Z^{**}=\frac{\Delta^{**2}}{\nu_0} \tag{9}$$

$$g(s)=SZ^{**}=g_1(s),\quad S=\frac{1}{\left(\dfrac{r}{L}\right)^{2j}}\frac{\rho_0\mu_0}{\rho_\mathrm{w}\mu_\mathrm{w}}\frac{\sigma_0 B_\mathrm{n}^2}{\rho_\mathrm{e}} \tag{10}$$

$$F_\mathrm{m}=2[\zeta-(2+H)f]+2gH_1 \tag{11}$$

$$H=\frac{\Delta^*}{\Delta^{**}},\quad \Delta^*(s)=\int_0^\infty\left(\frac{\rho_\mathrm{e}}{\rho}-\frac{u}{u_\mathrm{e}}\right)\mathrm{d}z,\quad \Delta^{**}(s)=\int_0^\infty\frac{u}{u_\mathrm{e}}\left(1-\frac{u}{u_\mathrm{e}}\right)\mathrm{d}z \tag{12}$$

$$H_1=\frac{\Delta_1^{**}}{\Delta^{**}},\quad \Delta_1^{**}(s)=\int_0^\infty\frac{u}{u_\mathrm{e}}\left(1-\frac{u}{u_\mathrm{e}}\right)\frac{\rho_\mathrm{e}}{\rho}\mathrm{d}z \tag{13}$$

$$\zeta=\left[\frac{\partial\left(\dfrac{u}{u_\mathrm{e}}\right)}{\partial\left(\dfrac{z}{\Delta^{**}}\right)}\right]_{z=0} \tag{14}$$

Note that all the expressions for (9)-(14) for $j=0$ are identical with the corresponding expressions for the planar boundary layer [20].

Transformation of the equations

Analogous to the already solved problems of the fluid flow in the boundary layer, a stream function $\psi(s, z)$ is introduced by the relations:

$$u=\frac{\partial\psi}{\partial z},\quad \tilde{v}=\frac{1}{\left(\dfrac{r}{L}\right)^{2j}}\frac{\rho_0\mu_0}{\rho_\mathrm{w}\mu_\mathrm{w}}\left[u\frac{\partial z}{\partial x}+v\frac{\rho}{\rho_0}\left(\frac{r}{L}\right)^j\right]=-\frac{\rho\psi}{\rho s} \tag{15}$$

which follow from the continuity equation.

Using the new variables (7) and introducing the stream function $\psi(s, z)$ by the relations (15), the governing equation system (2')-(5') is brought to this form:

$$\frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial s\partial z}-\frac{\partial\psi}{\partial s}\frac{\partial^2\psi}{\partial z^2}=\frac{\rho_\mathrm{e}}{\rho}u_\mathrm{e}u_\mathrm{e}'+\nu_0\frac{\partial}{\partial z}\left(Q\frac{\partial^2\psi}{\partial z^2}\right)-\frac{1}{\left(\dfrac{r}{L}\right)^{2j}}\frac{\rho_0\mu_0}{\rho_\mathrm{w}\mu_\mathrm{w}}\frac{\sigma B_\mathrm{m}^2}{\rho}\frac{\partial\psi}{\partial z} \tag{16}$$

$$\frac{\partial\psi}{\partial z}\frac{\partial h}{\partial s}-\frac{\partial\psi}{\partial s}\frac{\partial h}{\partial z}=-\frac{\rho_\mathrm{e}}{\rho}u_\mathrm{e}u_\mathrm{e}'\frac{\partial\psi}{\partial z}+\nu_0 Q\left(\frac{\partial^2\psi}{\partial z^2}\right)^2+\nu_0\frac{\partial}{\partial z}\left(\frac{Q}{\mathrm{Pr}}\frac{\partial h}{\partial z}\right)+$$
$$+\frac{1}{\left(\dfrac{r}{L}\right)^{2j}}\frac{\rho_0\mu_0}{\rho_\mathrm{w}\mu_\mathrm{w}}\frac{\sigma B_\mathrm{m}^2}{\rho}\left(\frac{\partial\psi}{\partial z}\right)^2 \tag{17}$$

$$\psi=0,\quad \frac{\partial\psi}{\partial z}=0,\quad h=h_{\mathrm{w}}=\mathrm{const.}\quad \text{for}\quad z=0$$
$$\frac{\partial\psi}{\partial z}\to u_{\mathrm{e}}(s),\quad h\to h_{\mathrm{e}}(s)\quad \text{for}\quad z\to\infty \tag{18}$$

In eqs. (16) and (17), the non-dimensional function Q is determined as:

$$Q=\frac{\rho\mu}{\rho_{\mathrm{w}}\mu_{\mathrm{w}}}=Q(s,z) \tag{19}$$

For $j = 0$, eqs. (16) and (17) are also identical to the corresponding equations for the planar ionized gas flow.

A new change of variables.
Introduction of functions $\Phi(s, \eta)$ and $B(s)$

In accordance with Saljnikov's version of the general similarity method, a new change of variables is introduced using the expressions:

$$s=s,\quad \eta(s,z)=\frac{u_{\mathrm{e}}^{b/2}}{K(s)}z,\quad \psi(s,z)=u_{\mathrm{e}}^{b/2}K(s)\Phi(s,\eta),\quad h(s,z)=h_1\bar{h}(s,\eta),\quad h_1=\mathrm{const.} \tag{20}$$

$$K(s)=\sqrt{av_0\int_0^s u_{\mathrm{e}}^{b-1}\mathrm{d}s},\quad a,b=\mathrm{const.} \tag{21}$$

Here the following notation is used, $\eta(s, z)$ – a newly introduced transversal variable, $\Phi(s, \eta)$ – non-dimensional stream function, $\bar{h}(s, \eta)$ – non-dimensional enthalpy, h_1 – total enthalpy in the outer flow, and a, b – arbitrary constants.

Therefore, based on expressions (20) and (21), certain quantities and characteristics of the boundary layer (9)-(14) can be written as:

$$\frac{u}{u_{\mathrm{e}}}=\frac{\partial\Phi}{\partial\eta} \tag{22}$$

$$K(s)=\frac{u_{\mathrm{e}}^{b/2}\Delta^{**}}{B},\quad B(s)=\int_0^\infty\frac{\partial\Phi}{\partial\eta}\left(1-\frac{\partial\Phi}{\partial\eta}\right)\mathrm{d}\eta \tag{23}$$

$$\frac{\Delta^{*}(s)}{\Delta^{**}(s)}=H=\frac{A(s)}{B(s)},\quad A(s)=\int_0^\infty\left(\frac{\rho_{\mathrm{e}}}{\rho}-\frac{\partial\Phi}{\partial\eta}\right)\mathrm{d}\eta \tag{24}$$

$$\frac{\Delta_1^{**}(s)}{\Delta^{**}(s)}=H_1=\frac{A_1(s)}{B(s)},\quad A_1(s)=\int_0^\infty\frac{\partial\Phi}{\partial\eta}\left(1-\frac{\partial\Phi}{\partial\eta}\right)\frac{\rho_{\mathrm{e}}}{\rho}\mathrm{d}\eta \tag{25}$$

$$\zeta=B\left(\frac{\partial^2\Phi}{\partial\eta^2}\right)_{\eta=0} \tag{26}$$

where the quantities A, A_1, and B are considered to be continuous functions of the variable s. Based on expressions (22)-(26), both the newly introduced variable η and the stream function $\psi(s, z)$ can be written in more suitable forms as:

$$s=s,\quad \eta(s,z)=\frac{B(s)}{\Delta^{**}(s)}z,\quad \psi(s,z)=\frac{u_{\mathrm{e}}(s)\Delta^{**}(s)}{B(s)}\Phi(s,\eta),\quad h(s,z)=h_1\bar{h}(s,\eta) \tag{27}$$

The newly introduced variables (20), *i. e.* (27), enable transformation of the governing equation system (16)-(18) into a form more suitable for further analysis. The transformed equation system, written using the new variables, takes the following form:

$$\frac{\partial}{\partial\eta}\left(Q\frac{\partial^2\Phi}{\partial\eta^2}\right)+\frac{aB^2+(2-b)f}{2B^2}\Phi\frac{\partial^2\Phi}{\partial\eta^2}+\frac{f}{B^2}\left[\frac{\rho_e}{\rho}-\left(\frac{\partial\Phi}{\partial\eta}\right)\right]^2-\frac{g}{B^2}\frac{\rho_e}{\rho}\left(1-\frac{\partial\Phi}{\partial\eta}\right)\frac{\partial\Phi}{\partial\eta}=$$
$$=\frac{u_e}{u_e'}\frac{f}{B^2}\left(\frac{\partial\Phi}{\partial\eta}\frac{\partial^2\Phi}{\partial s\partial\eta}-\frac{\partial\Phi}{\partial s}\frac{\partial^2\Phi}{\partial\eta^2}\right) \tag{28}$$

$$\frac{\partial}{\partial\eta}\left(\frac{Q}{\mathrm{Pr}}\frac{\partial\bar h}{\partial\eta}\right)+\frac{aB^2+(2-b)f}{2B^2}\Phi\frac{\partial\bar h}{\partial\eta}-\frac{2\kappa f}{B^2}\frac{\rho_e}{\rho}\frac{\partial\Phi}{\partial\eta}+2\kappa Q\left(\frac{\partial^2\Phi}{\partial\eta^2}\right)^2+$$
$$+\frac{2\kappa g}{B^2}\frac{\rho_e}{\rho}\left(1-\frac{\partial\Phi}{\partial\eta}\right)\left(\frac{\partial\Phi}{\partial\eta}\right)^2=\frac{u_e}{u_e'}\frac{f}{B^2}\left(\frac{\partial\Phi}{\partial\eta}\frac{\partial\bar h}{\partial s}-\frac{\partial\Phi}{\partial s}\frac{\partial\bar h}{\partial\eta}\right) \tag{29}$$

$$\Phi=0,\quad \frac{\partial\Phi}{\partial\eta}=0,\quad \bar h=\bar h_w=\text{const.}\quad \text{for}\quad \eta=0,$$
$$\frac{\partial\Phi}{\partial\eta}\to 1,\qquad \bar h\to\bar h_e=1-\kappa\quad \text{for}\quad \eta\to\infty \tag{30}$$

Note that eqs. (28) and (29) contain the factor u_e/u_e'. The solution of the system will depend on each concrete form of the law on variation of the velocity $u_e(s)$ on the outer edge of the boundary layer. Consequently, the obtained system (28)-(30) is not generalized in terms of Loitsianskii [1]. Therefore, the analysis has shown that the so-called generalized boundary layer equations cannot be obtained using the functions $\Phi(s,\eta)$ and $\bar h(s,\eta)$.

The density ρ_e/ρ ratio and the quantity κ are found in the energy equation and in the boundary conditions. As with the dissociated gas [3], the quantity κ stands for the local compressibility parameter which is defined as:

$$\kappa=f_0=\frac{u_e^2}{2h_1},\quad \frac{u_e}{u_e'}f_1\frac{d\kappa}{ds}=2\kappa f_1=\theta_0 \tag{31}$$

It represents an in advance given function of the variable s.

Generalized boundary layer equations

In order to bring the system (16)-(18) to a generalized form, the function Φ and the non-dimensional enthalpy $\bar h$ are introduced using the so-called general similarity transformations. In other words, in order to apply the general similarity method, the functions Φ and $\bar h$ are introduced using the expressions:

$$\psi(s,z)=\frac{u_e\Delta^{**}}{B}\Phi[\eta,\kappa,(f_k),(g_k)],\quad h(s,z)=h_1\bar h[\eta,\kappa,(f_k),(g_k)] \tag{32}$$

where (f_k) denotes a set of form parameters of Loitsianskii's type and (g_k) denotes a set of magnetic parameters [1, 7]. The introduced sets of similarity parameters are new independent variables (instead of the variable s) and, as with incompressible fluid, they are defined by the expressions:

$$f_k(s)=u_e^{k-1}u_e^{(k)}Z^{**k},\ (k=1,2,3,\ldots) \tag{33}$$

$$g_k(s)=u_e^{k-1}S^{(k-1)}Z^{**k} \tag{34}$$

The first parameters ($k=1$) of the sets (33) and (34) represent the earlier defined form parameter $f_1=u_e'Z^{**}=f$ and the magnetic parameter $g_1=SZ^{**}=g$.

Each set of parameters satisfies a corresponding recurrent simple differential equation of the form [1, 7]:

$$\frac{u_\mathrm{e}}{u'_\mathrm{e}} f_1 \frac{\mathrm{d}f_k}{\mathrm{d}s} = [(k-1)f_1 + kF_\mathrm{m}]f_k + f_{k+1} = \theta_k \tag{35}$$

$$\frac{u_\mathrm{e}}{u'_\mathrm{e}} f_1 \frac{\mathrm{d}g_k}{\mathrm{d}s} = [(k-1)f_1 + kF_\mathrm{m}]g_k + g_{k+1} = \gamma_k \tag{36}$$

Having applied the similarity transformations (32) and taking the expressions (27), (31), and (33)-(36) into consideration, the governing equation system (16)-(18) is finally transformed into the system:

$$\frac{\partial}{\partial\eta}\left(Q\frac{\rho^2\Phi}{\partial\eta^2}\right) + \frac{aB^2 + (2-b)f_1}{2B^2}\Phi\frac{\partial^2\Phi}{\partial\eta^2} + \frac{f_1}{B^2}\left[\frac{\rho_\mathrm{e}}{\rho} - \left(\frac{\partial\Phi}{\partial\eta}\right)^2\right] - \frac{g_1}{B^2}\frac{\rho_\mathrm{e}}{\rho}\left(1-\frac{\partial\Phi}{\partial\eta}\right)\frac{\partial\Phi}{\partial\eta} =$$

$$= \frac{1}{B^2}\left[\sum_{k=1}^{\infty}\theta_k\left(\frac{\partial\Phi}{\partial\eta}\frac{\partial^2\Phi}{\partial\eta\partial f_k} - \frac{\partial\Phi}{\partial f_k}\frac{\partial^2\Phi}{\partial\eta^2}\right) + \sum_{k=0}^{\infty}\gamma_k\left(\frac{\partial\Phi}{\partial\eta}\frac{\partial^2\Phi}{\partial\eta\partial g_k} - \frac{\partial\Phi}{\partial g_k}\frac{\partial^2\Phi}{\partial\eta^2}\right)\right] \tag{37}$$

$$\frac{\partial}{\partial\eta}\left(\frac{Q}{\mathrm{Pr}}\frac{\partial\bar{h}}{\partial\eta}\right) + \frac{aB^2 + (2-b)f_1}{2B^2}\Phi\frac{\partial\bar{h}}{\partial\eta} - \frac{2\kappa f_1}{B^2}\frac{\rho_\mathrm{e}}{\rho}\frac{\partial\Phi}{\partial\eta} +$$

$$+2\kappa Q\left(\frac{\partial^2\Phi}{\partial\eta^2}\right)^2 + \frac{2\kappa g_1}{B^2}\frac{\rho_\mathrm{e}}{\rho}\left(1-\frac{\partial\Phi}{\partial\eta}\right)\left(\frac{\partial\Phi}{\partial\eta}\right)^2 =$$

$$= \frac{1}{B^2}\left[\sum_{k=1}^{\infty}\theta_k\left(\frac{\partial\Phi}{\partial\eta}\frac{\partial\bar{h}}{\partial g_\mathrm{k}} - \frac{\partial\Phi}{\partial g_\mathrm{k}}\frac{\partial\bar{h}}{\partial\eta}\right) + \sum_{k=0}^{\infty}\gamma_k\left(\frac{\partial\Phi}{\partial\eta}\frac{\partial\bar{h}}{\partial f_\mathrm{k}} - \frac{\partial\Phi}{\partial f_\mathrm{k}}\frac{\partial\bar{h}}{\partial\eta}\right)\right] \tag{38}$$

$$\begin{gathered}\Phi = 0, \quad \frac{\partial\Phi}{\partial\eta} = 0, \quad \bar{h} = \bar{h}_\mathrm{w} = \text{const.} \quad \text{for} \quad \eta = 0 \\ \frac{\partial\Phi}{\partial\eta} \to 1, \quad \bar{h} \to \bar{h}_\mathrm{e}(s) = 1-\kappa \quad \text{for} \quad \eta = \infty\end{gathered} \tag{39}$$

In the system (37)-(39), distribution of the velocity $u_\mathrm{e}(s)$ at the outer edge of the boundary layer does not figure explicitly. In that sense, the system is generalized. Therefore, the system (37)-(39) is a general mathematical model of the ionized gas flow in the boundary layer adjacent to the bodies of revolution in the case when the variable electroconductivity is defined by the law (1). For $j = 0$, the system (37)-(39) is identical to the corresponding system for the planar ionized gas flow [20].

Since a numerical solution of the obtained equation system is practically impossible, the system is solved in the so-called n-parametric localized approximation. Here, it is solved in a three-parametric ($\kappa = f_0 \neq 0, f_1 = f \neq 0, g_1 = g \neq 0, f_\mathrm{k} = g_\mathrm{k} = 0$ for $k \geq 2$) twice localized ($\partial/\partial\kappa = 0$, $\partial/\partial g_1 = 0$) approximation. Thus, the system (37)-(39) is significantly simplified and it comes down to:

$$\frac{\partial}{\partial\eta}\left(Q\frac{\partial^2\Phi}{\partial\eta^2}\right) + \frac{aB^2 + (2-b)f_1}{2B^2}\Phi\frac{\partial^2\Phi}{\partial\eta^2} + \frac{f_1}{B^2}\left[\frac{\rho_\mathrm{e}}{\rho} - \left(\frac{\partial\Phi}{\partial\eta}\right)^2\right] - \frac{g_1}{B^2}\frac{\rho_\mathrm{e}}{\rho}\left(1-\frac{\partial\Phi}{\partial\eta}\right)\frac{\partial\Phi}{\partial\eta} =$$

$$= \frac{F_\mathrm{m}f_1}{B^2}\left(\frac{\partial\Phi}{\partial\eta}\frac{\partial^2\Phi}{\partial\eta\partial f_1} - \frac{\partial\Phi}{\partial f_1}\frac{\partial^2\Phi}{\partial\eta^2}\right) \tag{40}$$

$$\frac{\partial}{\partial\eta}\left(\frac{Q}{\Pr}\frac{\partial\bar{h}}{\partial\eta}\right)+\frac{aB^2+(2-b)f_1}{2B^2}\Phi\frac{\partial\bar{h}}{\partial\eta}-\frac{2\kappa f_1}{B^2}\frac{\rho_e}{\rho}\frac{\partial\Phi}{\partial\eta}+2\kappa Q\left(\frac{\partial^2\Phi}{\partial\eta^2}\right)^2+$$
$$+\frac{2\kappa g_1}{B^2}\frac{\rho_e}{\rho}\left(1-\frac{\partial\Phi}{\partial\eta}\right)\left(\frac{\partial\Phi}{\partial\eta}\right)^2=\frac{F_m f_1}{B^2}\left(\frac{\partial\Phi}{\partial\eta}\frac{\partial\bar{h}}{\partial f_1}-\frac{\partial\Phi}{\partial f_1}\frac{\partial\bar{h}}{\partial\eta}\right) \tag{41}$$

$$\Phi=0,\quad \frac{\partial\Phi}{\partial\eta}=0,\quad \bar{h}=\bar{h}_w\text{const.}\quad\text{for}\quad \eta=0,$$
$$\frac{\partial\Phi}{\partial\eta}\to 1,\quad \bar{h}\to\bar{h}_e(s)=1-\kappa\quad\text{for}\quad\eta\to\infty \tag{42}$$
$$[\Phi=\Phi^{(1)}(\eta,\kappa,f_1,g_1),\quad \bar{h}=\bar{h}^{(1)}(\eta,\kappa,f_1,g_1)]$$

In the system (40)-(42), based on (9)-(14):

$$F_m=2[\zeta-(2+H)f_1]+2g_1H_1 \tag{43}$$

The system of generalized eqs. (37)-(39) is an approximate mathematical model of the ionized gas flow in the boundary layer adjacent to the bodies of revolution. Due to the performed localization, the system is solved for in advance given values of the parameters $\kappa=f_0$ and g_1.

Again, note that the equations of the system (40)-(42) are formally the same as the corresponding equations for the planar ionized gas flow. For $j=0$ these equations are identical [20].

Numerical solution

The system of differential partial equations of the third order (40)-(42) is numerically solved after the order of the dynamic equation has been reduced by the change:

$$\frac{u}{u_e}=\frac{\partial\Phi}{\partial\eta}=\varphi(\eta,\kappa,f_1,g_1) \tag{44}$$

Based on (44), the equation system for numerical iteration is finally brought to:

$$\frac{\partial}{\partial\eta}\left(Q\frac{\partial\varphi}{\partial\eta}\right)+\frac{aB^2+(2-b)f_1}{2B^2}\Phi\frac{\partial\varphi}{\partial\eta}+\frac{f_1}{B^2}\left(\frac{\rho_e}{\rho}-\varphi^2\right)-\frac{g_1}{B^2}\frac{\rho_e}{\rho}(1-\varphi)\varphi=$$
$$=\frac{F_m f_1}{B^2}\left(\varphi\frac{\partial\varphi}{\partial f_1}-\frac{\partial\Phi}{\partial f_1}\frac{\partial\varphi}{\partial\eta_1}\right) \tag{45}$$

$$\frac{\partial}{\partial\eta}\left(\frac{Q}{\Pr}\frac{\partial\bar{h}}{\partial\eta}\right)+\frac{aB^2+(2-b)f_1}{2B^2}\Phi\frac{\partial\bar{h}}{\partial\eta}-\frac{2\kappa f_1}{B^2}\frac{\rho_e}{\rho}\varphi+2\kappa Q\left(\frac{\partial\varphi}{\partial\eta}\right)^2+\frac{2\kappa g_1}{B^2}\frac{\rho_e}{\rho}(1-\varphi)\varphi^2=$$
$$=\frac{F_m f_1}{B^2}\left(\varphi\frac{\partial\bar{h}}{\partial f_1}-\frac{\partial\Phi}{\partial f_1}\frac{\partial\bar{h}}{\partial\eta}\right) \tag{46}$$

$$\begin{gathered}\varphi=0,\quad \Phi=0,\quad \bar{h}=\bar{h}_w=\text{const.}\quad\text{for}\quad\eta=0\\ \varphi\to 1,\quad \bar{h}\to\bar{h}_e(s)=1-\kappa\quad\text{for}\quad\eta\to\infty\end{gathered} \tag{47}$$

For the function Q and the density ratio ρ_e/ρ which figure in the system (45)-(47), the following approximate dependences have been adopted [3]:

$$Q = Q(\bar{h}) \approx \sqrt[3]{\frac{\bar{h}_{\mathrm{w}}}{\bar{h}}}, \quad \frac{\rho_{\mathrm{e}}}{\rho} \approx \frac{\bar{h}}{1-\kappa} \tag{48}$$

It is obvious that the accurate laws on distribution of these quantities can be found only by detailed analysis (using thermodynamic tables for air) but this is not a primary objective of our investigation. Since Pr for air "depends negligibly on the temperature" [3, 7], the eqs. (45)-(47) are solved for a constant value of Pr = 0.712. For the constants a and b, the usual values are adopted [2]: a = 0.4408 and b = 5.7140.

The system of the conjugated partial differential equations (45)-(47) is numerically solved by the finite differences method – using the passage method. The boundary layer area is replaced by a planar integration grid, so the values of the functions φ, Φ, and $\bar{h}$ are calculated at discrete points of each calculating layer of this grid. A concrete numerical solution is performed using a programme written in *FORTRAN*, based on the one used in the paper [2]. Since the equation system is non-linear, it is solved by an iterative procedure. For the characteristic functions B, Q, and F_{m} at zero iteration, the usual values have been adopted [2].

Results

The system (45)-(47) is solved for each cross-section of the boundary layer using the computer programme. Solutions are obtained in tabular form. Only some of the obtained results are given here in the form of diagrams.

Figure 2 shows the diagram of the non-dimensional velocity $u/u_{\mathrm{e}} = \partial\Phi/\partial\eta$ at three cross-sections of the boundary layer. The diagram in fig. 3 presents the distribution of the non-dimensional enthalpy $\bar{h}$ for three cross-sections of the boundary layer.

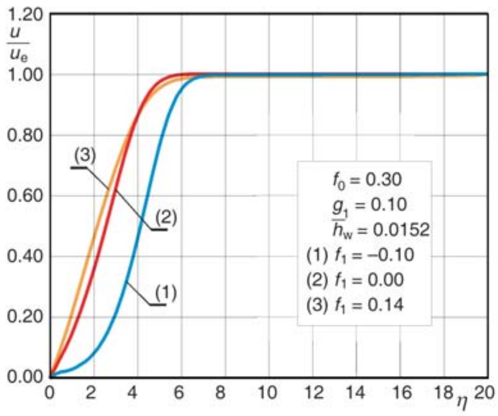

Figure 2. Diagram of the non-dimensional velocity u/u_{e}

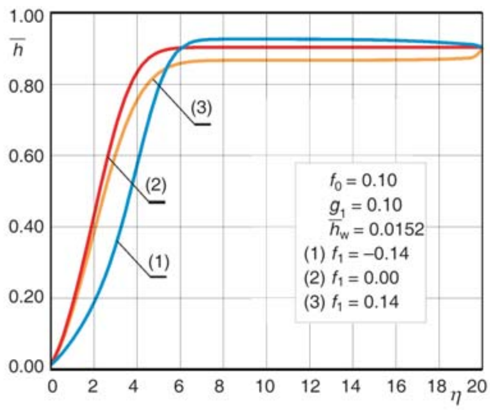

Figure 3. Diagram of the non-dimensional enthalpy $\bar{h}$

The influence of the compressibility parameter on the distribution of the non-dimensional enthalpy $\bar{h}$ is illustrated in fig. 4 which shows a diagram of the enthalpy $\bar{h}$ at one cross-section of the boundary layer (f_1 = 0.10) for three values of this parameter ($\kappa = f_0$ = 0.10; 0.15; 0.20). Figure 5 shows the distribution of the non-dimensional friction function $\zeta(f_1)$ in the boundary layer for three different values of the magnetic parameter. Figure 6 shows the distribution of the boundary layer characteristic function $B(f_1)$ for three values of the magnetic parameter g_1. Finally, fig. 7 shows the distribution of the boundary layer characteristic function $F_{\mathrm{m}}(f_1)$.

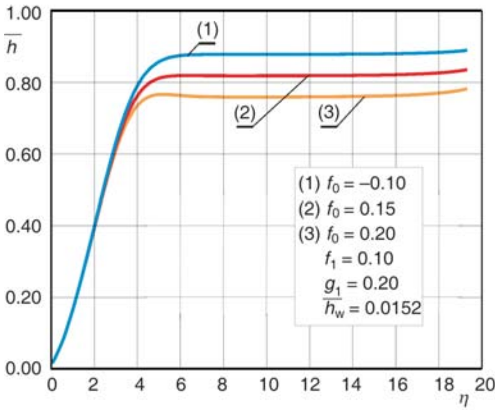

Figure 4. Diagram of the non-dimensional enthalpy $\bar{h}$ for different values of the parameter $\kappa = f_0$

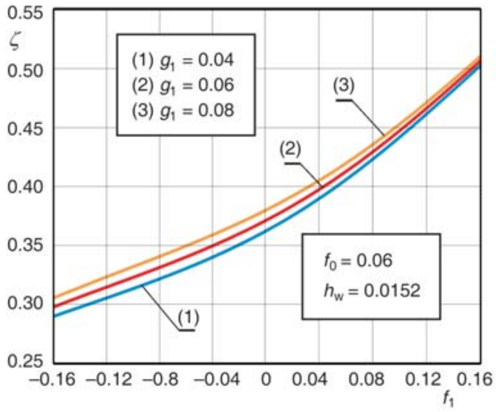

Figure 5. Distribution of the non-dimensional friction function ζ (f_1)

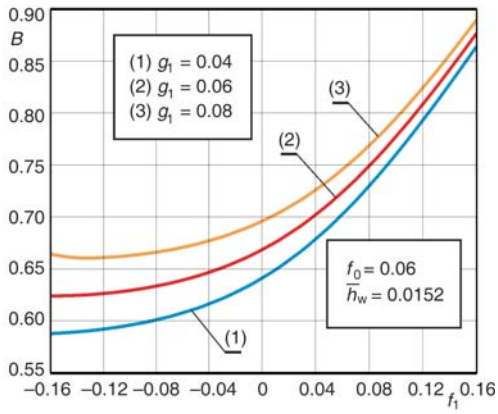

Figure 6. Distribution of the boundary layer characteristics $B(f_1)$

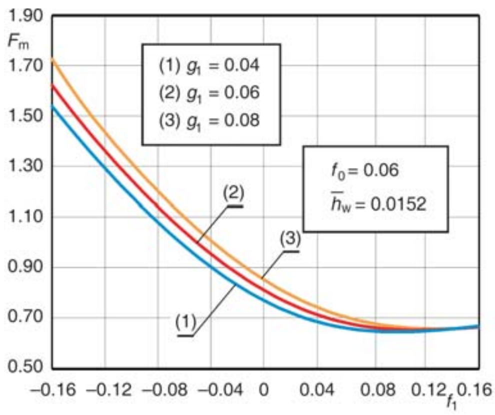

Figure 7. Distribution of the characteristic function F_m (f_1)

Conclusions

The results of our investigations yield two important general conclusions:

(1) Saljnikov's version of the general similarity method can be successfully applied to the studied problem of the axisymmetrical ionized gas flow in the boundary layer.

(2) Distributions of the physical (u/u_e, $\bar{h}, \zeta$, ...), and characteristic boundary layer quantities (B, F_m, ...) have the same behaviour as with other problems of dissociated and ionized gas flow in the boundary layer [3, 20].

Based on the results and the shown diagrams the following concrete conclusions can be drawn:

- The non-dimensional flow velocity u/u_e (fig. 2) converges very fast towards unity at certain cross-sections (conffuser and diffuser region) of the boundary layer. It is obvious that the variation of the form parameter has a small influence on convergence of the non-dimensional velocity at the outer edge of the boundary layer.

- The non-dimensional enthalpy $\bar{h} = h / h_1$ also converges very fast towards the value $\bar{h} = 1 - \kappa$ at the outer edge of the boundary layer (fig. 3).
- According to the diagram in the fig. 4, it can be concluded that the compressibility parameter has a great influence on the distribution of the non-dimensional enthalpy $\bar{h}$ in the boundary layer, which complies with the results for the dissociated gas [3] and ionized gas flow [20].
- The magnetic parameter g_1 has an influence on the non-dimensional friction function ζ (fig. 5) and has even a greater influence on the function B (fig. 6) and the characteristic function F_m (fig. 7). The increase in the value of the magnetic parameter brings about the increase in the value of the non-dimensional friction function ζ, which means that the separation of the boundary layer is postponed. These findings make a significant contribution to knowledge and understanding the ionized gas flow in the boundary layer.

Since the compressibility parameter has a great influence on the non-dimensional enthalpy [3, 21] (changing even the general behaviour of the distribution of the enthalpy $\bar{h}$), the boundary layer equations have to be solved in a three-parametric approximation without localization per the compressibility parameter. This would yield results that are more accurate. However, this kind of solution is fraught with considerable difficulties, mainly of mathematical and programming nature. This could be the subject of our further investigations.

Nomenclature

A, A_1, B – boundary layer characteristic, [–]
B_m – induction of outer magnetic field, [= $B_m(x)$], [Vsm^{-2}]
a, b – constants, [–]
c_p – specific heat of ionized gas at constant pressure, [Jkg^{-1}K^{-1}]
F_m – characteristic boundary layer function, [–]
f_1 – first form parameter, (= f), [–]
f_k – set of form parameters, [–]
g_1 – first magnetic parameter (= g), [–]
g_k – set of magnetic parameters, [–]
H_1, H – boundary layer characteristic, [–]
h – enthalpy, [Jkg^{-1}]
$\bar{h}$ – non-dimensional enthalpy, [–]
h_e – enthalpy at the outer edge of the boundary layer, [Jkg^{-1}]
h_w – enthalpy at the wall of the body within the fluid, [Jkg^{-1}]
h_1 – total enthalpy in the outer flow, [Jkg^{-1}]
j, k – number, [–]
Pr – Prandtl number (= $\mu c_p/\lambda$), [–]
p – pressure, [Pa]
Q – non-dimensional function, [–]
r – radius of the body of revolution in the meridian plane, [m]
s – new longitudinal variable, [m]
u – longitudinal projection of velocity in the boundary layer, [ms^{-1}]
u_e – velocity at the boundary layer outer edge, [ms^{-1}]
v – transversal projection of velocity in the boundary layer, [ms^{-1}]
x, y – longitudinal and transversal co-ordinate, [m]
Z^{**} – function, [s]
z – new transversal variable, [m]

Greek symbols

Δ^* – conditional displacement thickness, [m]
Δ^{**} – conditional momentum loss thickness, [m]
Δ_1^{**} – conditional thickness, [m]
δ – boundary layer thickness, [m]
ζ – non-dimensional friction function, [–]
η – non-dimensional transversal co-ordinate, [–]
κ – local compressibility parameter (= f_0), [–]
λ – thermal conductivity coefficient, [Wm^{-1}K^{-1}]
μ – dynamic viscosity, [Pa·s]
μ_0 – known values of dynamic viscosity of the ionized gas, [Pa·s]
μ_w – given distributions of dynamic viscosity at the wall of the body within the fluid, [Pa·s]
ν_0 – kinematic viscosity at a concrete point of the boundary layer, [m^{2}s^{-1}]
ρ – density of ionized gas, [kgm^{-3}]
ρ_e – ionized gas density at the outer edge of the boundary layer, [kgm^{-3}]
ρ_0 – known values of density of the ionized gas, [kgm^{-3}]
ρ_w – given distributions of density at the wall of the body within the fluid, [kgm^{-3}]
σ – electro-conductivity, [Nm^{3}V^{-2}s^{-1}]
Φ – non-dimensional stream function, [–]
ψ – steam function, [m^{2}s^{-1}]

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Paper submitted: December 5, 2011
Paper revised: April 2, 2012
Paper accepted: December 18, 2012